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EXTENDING LIPSCHITZ MAPS INTO $\mathcal{C}(K)$-SPACES

BY

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ABSTRACT

We show that if K is a compact metric space then $\mathcal{C}(K)$ is a 2-absolute Lipschitz retract. We then study the best Lipschitz extension constants for maps into $\mathcal{C}(K)$ from a given metric space M, extending recent results of Lancien and Randrianantoanina. They showed that a finite-dimensional normed space which is polyhedral has the isometric extension property for $\mathcal{C}(K)$-spaces; here we show that the same result holds for spaces with Gateaux smooth norm or of dimension two; a three-dimensional counterexample is also given. We also show that X is polyhedral if and only if every subset E of X has the universal isometric extension property for $\mathcal{C}(K)$-spaces. We also answer a question of Naor on the extension of Hölder continuous maps.

1. Introduction

Lipschitz extension of maps into $\mathcal{C}(K)$-spaces have been studied by a number of authors. The first results in this field are due to Lindenstrauss [10]. Lindenstrauss showed that if K is a compact metric space, then $\mathcal{C}(K)$ is an absolute Lipschitz retract. This implies that for a suitable constant λ if M is a metric space and E is a closed subset of M then every Lipschitz map $F_0\colon E \to \mathcal{C}(K)$ has an extension $F\colon M \to \mathcal{C}(K)$ with Lipschitz constant $\mathrm{Lip}(F) \le \lambda\,\mathrm{Lip}(F_0)$. Lindenstrauss's technique gave an estimate of $\lambda \le 20$. However c_0 is a 2-absolute Lipschitz retract, so that the corresponding result for c_0 works with constant 2. Note that, for K nonmetrizable one gets a similar extension result when E is separable. See the recent book of Benyamini and Lindenstrauss [1].

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Recently Lancien and Randrianantoanina [9] considered a related problem. They asked for conditions on M which guarantee extensions with $\lambda = 1$ (isometric case) or $\lambda = 1 + \epsilon$ (almost isometric case). They restricted $M = X$ to be a finite-dimensional normed space and showed that for the c_0-case one always has isometric extensions. However, for the $\mathcal{C}(K)$-case they showed that only $(1 + \epsilon)$-extensions are obtained in general and gave a four-dimensional counterexample to the isometric version. They showed, however, that for polyhedral spaces one always has an isometric extension.

Our first result in this paper improves Lindenstrauss's 1964 estimate by showing that $\mathcal{C}(K)$ is a 2-absolute retract for every compact metric space K.

We then develop necessary and sufficient conditions on a subset E of a metric space M for the existence of Lipschitz extensions of maps into c_0 or $\mathcal{C}(K)$ with prescribed Lipschitz constant (Theorems 4.1 and 4.2). These conditions permit us to determine the best constant for extensions, in principle. They also reveal a phenomenon already observed in [9]; that one can reduce to the case when K is the one point compactification of $\mathbb{N}$ (i.e. $\mathcal{C}(K) = c$). We then apply these conditions to the problems considered by Lancien and Randrianantoanina.

Let us introduce some definitions. We say that (E, M) has the **$\mathcal{C}$-IEP** (**$\mathcal{C}$-isometric extension property**) if for every Lipshitz map F_0: $E \to \mathcal{C}(K)$ (K compact metric) there is an extension F: $M \to \mathcal{C}(K)$ with $\mathrm{Lip}(F) = \mathrm{Lip}(F_0)$. We say that M has the **$\mathcal{C}$-IEP** if (E, M) has the $\mathcal{C}$-IEP for every subset E of M and the **$\mathcal{C}$-UIEP** (**universal isometric extension property**) if (M, M') has the $\mathcal{C}$-IEP, whenever $M \subset M'$.

Lancien and Randrianantoanina [9] showed that a finite-dimensional normed space X which is polyhedral has the $\mathcal{C}$-IEP; we give a different argument for this result and we extend it to show it also has the $\mathcal{C}$-UIEP (and hence so does every subset). We then show that X is polyhedral if and only if it has the $\mathcal{C}$-UIEP and the $\mathcal{C}$-IEP. To establish this we introduce a natural property for metric spaces which implies both $\mathcal{C}$-UIEP and $\mathcal{C}$-IEP. We say that a metric space M has the collinearity property of given $\epsilon > 0$ and an infinite subset A of M there are three points $x_1, x_2, x_3 \in A$ such that

$$d(x_1, x_3) > d(x_1, x_2) + d(x_2, x_3) - \epsilon.$$

For finite-dimensional normed spaces this property is equivalent to being polyhedral.

We then show that there are many other examples of finite-dimensional normed spaces with the $\mathcal{C}$-IEP, including any space with a Gateaux smooth norm. We also show that any 2-dimensional space has the $\mathcal{C}$-IEP, and show that a result of Lindenstrauss [11] implies the existence of a 3-dimensional space failing $\mathcal{C}$-IEP.

We conclude by considering Hölder extensions and give an example which answers a question of Naor [14]. This is an example of a metric space M with the $\mathcal{C}$-IEP but such that if $0 < \alpha < 1$ the space $M^\alpha = (M, d^\alpha)$ fails the $\mathcal{C}$-IEP. This means that isometric extensions are always possible for Lipschitz maps but not for Hölder continuous maps.

2. Preliminaries

Let (M, d) be a metric space. A subset A of M is called **metrically bounded** if for some (and hence for every) $x \in M$ we have $\sup_{a\in A} d(a, x) < \infty$. We will say that M has the **Heine-Borel property** if every metrically bounded subset is precompact (or totally bounded); this is equivalent to requiring that every metrically bounded sequence has a Cauchy subsequence. If M is complete it implies that M is locally compact but the converse is false.

Suppose (Y, d_1) is another metric space and $F\colon M \to Y$ is a Lipschitz map we denote by $\mathrm{Lip}(F)$ the Lipschitz constant of F, i.e., the least constant λ such that

$$d_1(F(x_1), F(x_2)) \leq \lambda d(x_1, x_2) \quad x_1, x_2 \in M.$$

Suppose $A \subset M$. We say that the pair (A, M) has the Lipschitz (λ, Y)**-extension property** (**Lipschitz** (λ, Y)**-EP**) if for every Lipschitz map $F_0\colon A \to Y$ there is a Lipschitz extension $F\colon M \to Y$ with $\mathrm{Lip}(F) \leq \lambda\,\mathrm{Lip}(F_0)$. In particular we say (A, M) has the **Lipschitz isometric** Y**-extension property** (**Lipschitz** Y**-IEP**) if it has the $(1, Y)$-EP, and the **Lipschitz almost isometric** Y**-extension property** (**Lipschitz** Y**-AIEP**) if it has the (λ, Y)-EP for every $\lambda > 1$.

We also say that M has the **Lipschitz** (λ, Y)**-extension property** (**Lipschitz** (λ, Y)**-EP**) [respectively, **Lipschitz isometric** Y**-extension property,** **(**Y**-IEP)**; respectively **Lipschitz almost isometric** Y**-extension property** (**Lipschitz** Y**-AIEP**)] if for for every subset A of M the pair (A, M) has the Lipschitz (λ, Y)-EP [respectively, Lipschitz Y-IEP; respectively, Lipschitz Y-AIEP]. We will be especially interested in the cases when Y is the Banach space c_0 or $\mathcal{C}(K)$ where K is a compact metric space.

Similarly we will say that M has the **Lipschitz universal** (λ, Y)**-extension property** (**Lipschitz** (λ, Y)**-UEP**) if whenever M is embedded in a metric space M' then the pair (M, M') has the Lipschitz (λ, Y)-EP. We similarly define the Lipschitz universal isometric and Lipschitz almost isometric Y-extension properties (Lipschitz Y-UIEP and Lipschitz Y-UAIEP).

It is sometimes natural to consider combinations of these properties. We will need the following

PROPOSITION 2.1: *Suppose M, Y are metric spaces and $\lambda \geq 1$. The following are equivalent:*

(i) *M has the Lipschitz (λ, Y)-EP and the Lipschitz (λ, Y)-UEP.*

(ii) *Every subset of M has the Lipschitz (λ, Y)-UEP.*

Proof: (ii) $\Rightarrow$ (i) is obvious. For (i)$\Rightarrow$ (ii) we observe that it suffices to argue that if E is a subset of M and E is simultaneously isometrically embedded in some other metric space (M', d'), we can form a pseudo-metric on $M \cup M'$ by defining

$$d(x, x') = \inf\{d(x, e) + d'(e, x') : e \in E\} \quad x \in M \setminus E, x' \in M' \setminus E.$$

If (ii) holds it follows that we can extend any Lipschitz map $F_0 \colon E \to Y$ to $F \colon M \cup M' \to Y$ with $\mathrm{Lip}(F) \leq \lambda \, \mathrm{Lip}(F_0)$ and then restrict it to M'. ∎

A is called a λ-Lipschitz retract of M if (A, M) has the (λ, Y)-extension property for every choice of Y. This is equivalent to the requirement that there is a Lipschitz retract $r \colon M \to A$ with $\mathrm{Lip}(r) \leq \lambda$. ($r$ is a retract if $r(a) = a$ for $a \in A$.) Y is called a **λ-absolute Lipschitz retract** if every pair (A, M) has the (λ, Y)-EP. It is well-known that $\mathbb{R}$ is a 1-absolute Lipschitz retract and hence that every (real) Banach space $\ell_\infty(S)$ is a 1-absolute Lipschitz retract. In contrast to the linear theory it is however true that c_0 is a 2-absolute Lipschitz retract ([10], [1]). It is also known that any $\mathcal{C}(K)$-space with K compact metric is a 20-absolute Lipschitz retract ([10], [1]). We will improve this estimate shortly.

Now suppose X, Y are Banach spaces and E is a closed subspace of X. We say that (E, X) has the **linear (λ, Y)-extension property (linear (λ, Y)-EP)** if for every bounded linear operator $T_0 \colon E \to Y$ there is a bounded linear extension $T \colon X \to Y$ with $\|T\| \leq \lambda \|T_0\|$. We would like to take the opportunity to relate the linear and nonlinear theories by the following simple Lemma:

LEMMA 2.2: *Suppose X and Y are Banach spaces and suppose E is a closed subspace of X of co-dimension one. Suppose (E, X) has the Lipschitz (λ, Y)-EP. Then (E, X) has the linear (λ, Y)-EP.*

Proof: Suppose $T_0 \colon E \to Y$ is a bounded linear operator. Then there is a Lipschitz extension $F \colon X \to Y$ with $\mathrm{Lip}(F) \leq \lambda \|T_0\|$. Pick any $x_0 \in X \setminus E$ and extend T_0 to a linear map T: $X \to Y$ such that $Tx_0 = F(x_0)$. Then for any $e \in E$

$$\|T(e+x_0)\| = \|F(x_0) - F(-e)\| \le \lambda \|T_0\| \|x_0 + e\|$$

and it follows trivially that $\|T\| \le \lambda \|T_0\|$. ∎

Finally let us note that if $M = (M, d)$ is a metric space, then M^α denotes the metric space (M, d^α) for $0 < \alpha < 1$. If $M_j = (M_j, d_j)$ for $j = 1, 2, \ldots, n$ are metric spaces then $(\sum_{j=1}^n M_j)_{\ell_1} = M_1 \oplus_1 \cdots \oplus_1 M_n$ denotes the metric space $M_1 \times \cdots \times M_n$ with the metric

$$d((x_j)_{j=1}^n, (y_j)_{j=1}^n) = \sum_{j=1}^n d_j(x_j, y_j)$$

and $(\sum_{j=1}^n M_j)_{\ell_\infty} = M_1 \oplus_\infty \cdots \oplus_\infty M_n$ denotes the metric space $M_1 \times \cdots \times M_n$ with the metric

$$d((x_j)_{j=1}^n, (y_j)_{j=1}^n) = \max_{1 \le j \le n} d_j(x_j, y_j).$$

3. $C(K)$-extensions

We first state a simple well-known consequence of the classical Miljutin Lemma. We denote by Δ the Cantor set $\{-1, +1\}^{\mathbb{N}}$.

PROPOSITION 3.1: *Let K be any compact metric space. Then there exist positive contractive operators S: $\mathcal{C}(K) \to \mathcal{C}(\Delta)$ and R: $\mathcal{C}(\Delta) \to \mathcal{C}(K)$ such that $R1 = 1, S1 = 1$ and $RS = I_{\mathcal{C}(K)}$.*

Proof: K can be embedded into the Hilbert cube $[0,1]^{\mathbb{N}}$. Let φ: $K \to [0,1]^{\mathbb{N}}$ be such an embedding. A theorem of Borsuk [2] implies that there is a positive contractive operator A: $\mathcal{C}(K) \to \mathcal{C}([0,1]^{\mathbb{N}})$ with $A1 = 1$ and such that $(Af) \circ \varphi = f$. On the other hand, Miljutin [13] shows that there is a continuous surjection π: $\Delta \to [0,1]^{\mathbb{N}}$ and a positive contractive operator B: $\mathcal{C}(\Delta) \to \mathcal{C}([0,1]^{\mathbb{N}})$ so that $B(f \circ \pi) = f$. Now define $Sf = (Af) \circ \pi$ and $Rf = (Bf) \circ \varphi$. ∎

Let K be a compact Hausdorff space. We let $\ell_\infty(K)$ be the metric space of all bounded functions on K with the usual sup norm. Let $\mathcal{U}(K)$ and $\mathcal{L}(K)$ be respectively the subsets of $\ell_\infty(K)$ of all upper-semi-continuous functions and of all lower-semi-continuous functions. Thus, $\mathcal{C}(K) = \mathcal{U}(K) \cap \mathcal{L}(K)$.

Now let us specialize to the case of the Cantor set. We define $\mathcal{S}(\Delta)$ to be the subset of $\mathcal{U}(\Delta) \oplus_\infty \mathcal{L}(\Delta)$ consisting of all pairs (u, v) such that $u \le v$. Let

$\operatorname{diag}\mathcal{S}(\Delta) = \{(f,f) : f \in \mathcal{C}(\Delta)\}$ to be the canonical image of $\mathcal{C}(\Delta)$ in $\mathcal{S}(\Delta)$. We also define a partial order on $\mathcal{S}(\Delta)$ by $(u_1,v_1) \preceq (u_2,v_2)$ if $u_1 \geq u_2$ and $v_1 \leq v_2$.

THEOREM 3.2: *There is a 1-Lipschitz retract* θ: $\mathcal{S}(\Delta) \to \operatorname{diag}\mathcal{S}(\Delta)$ *such that* $\theta(u,v) \preceq (u,v)$ *for all* $(u,v) \in \mathcal{S}(\Delta)$.

Proof: Let $\mathcal{V}_m$ be the family of 2^m clopen sets of the form

$$\Delta_{\epsilon_1,\ldots,\epsilon_m} = \{s = (s_k)_{k=1}^{\infty} : s_j = \epsilon_j, 1 \leq j \leq m\}.$$

Let $\mathcal{C}_m(\Delta)$ be the subset of $\mathcal{C}(\Delta)$ of all functions constant on each set $A \in \mathcal{V}_m$; this set is isometric to $\ell_\infty^{2^m}$. For each $h \in \ell_\infty(K)$ define

$$\alpha_m h = \sum_{A \in \mathcal{V}_m} (\inf_A h)\chi_A.$$

Thus α_m: $\ell_\infty(\Delta) \to \mathcal{C}_m(\Delta)$ is a 1-Lipschitz map.

Next, let $\mathcal{A}_m$ be defined for $m = 1, 2, \ldots,$ by

$$\mathcal{A}_m = \{(u,v) \in \mathcal{S}(\Delta) : \alpha_m v \geq u\}.$$

LEMMA 3.3: *There is a 1-Lipschitz map* ψ_m: $\mathcal{S}(\Delta) \to \mathcal{S}(\Delta)$ *such that:*

(i) ψ_m *is a retraction of* $\mathcal{A}_m$ *onto* $\operatorname{diag} S(\Delta) \cap \mathcal{A}_m$.

(ii) $\psi_m(u,v) \preceq (u,v)$ *for all* $(u,v) \in \mathcal{S}(\Delta)$.

(iii) *If* $n \geq m$ *then* $\psi_m(\mathcal{A}_n) \subset \mathcal{A}_n$.

Note here that $\operatorname{diag}\mathcal{S}(\Delta) \cap \mathcal{A}_m = \{(f,f) : f \in \mathcal{C}_m(\Delta)\}$.

Proof of the Lemma: Let us define σ_m: $\mathcal{S}(\Delta) \to \mathcal{C}_m(\Delta)$ by

$$\sigma_m(u,v) = \inf_{(u',v') \in \mathcal{A}_m} (\alpha_m v' + \max(\|u-u'\|_\infty, \|v-v'\|_\infty)).$$

Here the infimum is taken pointwise, and it is clear that σ_m is a 1-Lipschitz map. Note that for all $(u',v') \in \mathcal{A}_m$ we have

$$\alpha_m v \leq \alpha_m v' + \|v-v'\|_\infty$$

and

$$u \leq u' + \|u-u'\|_\infty \leq \alpha_m v' + \|u-u'\|_\infty$$

so that we have the properties:

$$\sigma_m(u,v) \geq \max(u, \alpha_m v) \quad (u,v) \in \mathcal{S}(\Delta). \tag{3.1}$$

and

$$\sigma_m(u,v) = \alpha_m v \quad (u,v) \in \mathcal{A}_m. \tag{3.2}$$

Similarly, we define $\tau_m\colon \mathcal{S}(\Delta) \to \mathcal{C}_m(\Delta)$ by

$$\tau_m(u,v) = \sup_{(u',v')\in\mathcal{A}_m} (\alpha_m v' - \max(\|u-u'\|_\infty, \|v-v'\|_\infty)).$$

Then τ_m is also 1-Lipschitz and we have

$$\tau_m(u,v) \le \alpha_m v \quad (u,v) \in \mathcal{S}(\Delta) \tag{3.3}$$

and

$$\tau_m(u,v) = \alpha_m v \quad (u,v) \in \mathcal{A}_m. \tag{3.4}$$

Thus σ_m and τ_m are simply the maximal and minimal 1-Lipschitz extensions of the map $(u,v) \to \alpha_m v$ from $\mathcal{A}_m$ into $\mathcal{C}_m(\Delta)$.

Define

$$\psi_m(u,v) = (\max(u, \tau_m(u,v)), \min(v, \sigma_m(u,v))) \quad (u,v) \in \mathcal{S}(\Delta).$$

The first component is clearly in $\mathcal{U}(\Delta)$ and the second component is in $\mathcal{L}(\Delta)$. Furthermore, we have by (3.1) and (3.3), that

$$\max(u, \tau_m(u,v)) \le \min(v, \sigma_m(u,v))).$$

Hence ψ_m maps $\mathcal{S}(\Delta)$ to itself and (ii) holds.

If $(u,v) \in \mathcal{A}_n$ where $n \ge m$ then $\sigma_m(u,v)$ and $\tau_m(u,v)$ are constant on each $A \in \mathcal{V}_n$. Hence,

$$\alpha_n(\min(v, \sigma_m(u,v)) = \min(\alpha_n v, \sigma_m(u,v)) \ge \max(u, \tau_m(u,v)).$$

Thus if $n \ge m$ we have $\psi_m(\mathcal{A}_n) \subset \mathcal{A}_n$, i.e., (iii) holds.

We also have

$$|\max(u, \tau_m(u,v)) - \max(u', \tau_m(u',v'))| \le \max(\|u-u'\|_\infty, \|v-v'\|_\infty)$$

and

$$|\min(v, \sigma_m(u,v)) - \min(v', \sigma_m(u',v'))| \le \max(\|u-u'\|_\infty, \|v-v'\|_\infty)$$

so that ψ_m is 1-Lipschitz.

If $(u,v) \in \mathcal{A}_m$ then $\psi_m(u,v) = (\alpha_m v, \alpha_m v)$ so that (i) holds. ∎

To complete the proof of the theorem, we will show that

$$\theta_\Delta(u,v) = \lim_{m\to\infty} \psi_m \circ \psi_{m-1} \circ \cdots \circ \psi_1(u,v)$$

defines the required map. Notice that the right-hand side converges and is eventually constant if $(u,v) \in \bigcup_{m\geq 1} \mathcal{A}_m$. To prove pointwise convergence for all (u,v) it will suffice (since all the maps are 1-Lipschitz) to show that $\bigcup_{m\geq 1} \mathcal{A}_m$ is dense in $\mathcal{S}(\Delta)$. In fact, the set of (u,v) so that $\inf(v-u) > 0$ is contained in the union. Indeed, $\alpha_m v - u$ is an increasing sequence of lower-semi-continuous functions which converges pointwise to $v-u$. By Dini's theorem there exists m so that $\alpha_m v - u \geq 0$ everywhere i.e. $(u,v) \in \mathcal{A}_m$. Since the set of (u,v) with $\inf(v-u) > 0$ is trivially dense we have convergence.

Obviously θ_Δ is contractive and $\theta_\Delta(u,v) \preceq (u,v)$. Furthermore since $\bigcup_{m\geq 1} \mathcal{A}_m$ is mapped into $\operatorname{diag} \mathcal{S}(\Delta)$ it is clear that θ_Δ satisfies all our conditions. ∎

The following theorem will be basic for our future considerations:

THEOREM 3.4: *Let K be a compact metric space and M any metric space. Suppose $F_l\colon M \to \mathcal{L}(K)$ and $F_u\colon M \to \mathcal{U}(K)$ are two Lipschitz maps such that*

$$F_u(x) \leq F_l(x) \quad x \in M.$$

Then there is a Lipschitz map $F\colon M \to \mathcal{C}(K)$ such that

$$F_u(x) \leq F(x) \leq F_l(x) \quad x \in M$$

and $\operatorname{Lip}(f) \leq \max(\operatorname{Lip}(F_l), \operatorname{Lip}(F_u))$.

Proof: Let us prove this for the special case when $K = \Delta$. Indeed in this case we simply define $F(x)$ by $(F(x), F(x)) = \theta_\Delta(F_u(x), F_l(x))$, where θ_Δ is given by Theorem 3.2.

Now let K be an arbitrary compact metric space. Let R, S be the operators given by Proposition 3.1. We can regard $\mathcal{U}(K)$ and $\mathcal{L}(K)$ as subsets of $\mathcal{C}(K)^{**}$ via the formula

$$\langle \mu, f\rangle = \int f\, d\mu \quad \mu \in \mathcal{M}(K).$$

Assume the result is known for Δ. Then given F_u, F_l as above we define $G_u = S^{**}F_u$ and $G_l = S^{**}F_l$. We claim that if $x \in M$, $G_u(x) \in \mathcal{U}(\Delta)$. Indeed, take any bounded sequence $(f_n)_{n=1}^\infty$ in $\mathcal{C}(K)$ so that $f_n(t) \downarrow F_u(x)(t)$ for $t \in K$. Then

$Sf_n(t) \downarrow g(t)$ for some $g \in \mathcal{U}(\Delta)$. However it is clear that $g = S^{**}F_u(x)$ since for any $\mu \in \mathcal{M}(\Delta)$ we have

$$\int g d\mu = \lim_{n\to\infty} \int Sf_n d\mu = \int F_u(x) d(S^*\mu).$$

Similarly we can argue that $G_l(x) \in \mathcal{L}(\Delta)$. Clearly, $\mathrm{Lip}(G_u) \leq \mathrm{Lip}(F_u)$ and $\mathrm{Lip}(G_l) \leq \mathrm{Lip}(F_l)$. It then follows we can find a Lipschitz map $G\colon M \to \mathcal{C}(\Delta)$ with

$$G_u(x) \leq G(x) \leq G_l(x) \quad x \in M$$

and $\mathrm{Lip}(G) \leq \max(\mathrm{Lip}(F_u), \mathrm{Lip}(F_l))$. Define $F(x) = R(G(x))$ and we are done. ∎

The following theorem improves the result of Lindenstrauss [10], who proved the same result with constant 20. He also obtained the constant 2 but only for extending maps to finitely many additional points.

Theorem 3.5: *If K is a compact metric space, then $\mathcal{C}(K)$ is a 2-absolute Lipschitz retract.*

Proof: It is enough to produce a 2-Lipschitz retract from $\ell_\infty(K)$ onto $\mathcal{C}(K)$. For any $f \in \ell_\infty(K)$, define Lf, Uf to be its lower-semi-continuous and upper-semi-continuous regularizations i.e.,

$$Lf(s) = \liminf_{t\to s} f(t), \quad Uf(s) = \limsup_{t\to s} f(t) \quad s \in K.$$

Then $\mathrm{Lip}(L), \mathrm{Lip}(U) = 1$. Now $Lf \leq f \leq Uf$ and in fact

$$Uf \leq Lf + 2d(f, \mathcal{C}(K)).$$

Hence, if we define

$$F_u(f) = Uf - d(f, \mathcal{C}(K)), \quad F_l(f) = Lf + d(f, \mathcal{C}(K)),$$

then $\mathrm{Lip}(F_u), \mathrm{Lip}(F_l) \leq 2$ and the hypotheses of Theorem 3.4 are satisfied. Hence we obtain a Lipschitz map $F\colon \ell_\infty(K) \to \mathcal{C}(K)$ with $\mathrm{Lip}(F) \leq 2$ and

$$Uf - d(f, \mathcal{C}(K)) \leq F(f) \leq Lf + d(f, \mathcal{C}(K)).$$

Clearly F is our desired Lipschitz retract. ∎

4. A criterion for the existence of extensions

In this section we develop criteria to determine whether (E, M) has the (λ, c_0)-extension property or the $(\lambda, \mathcal{C}(K))$-extension property. This is based on ideas in [9].

THEOREM 4.1: *Suppose M is a metric space and E is a separable subset of M. Then the following statements are equivalent:*

(i) *(E, M) has the Lipschitz (λ, c_0)-EP.*

(ii) *For every $a \in M \setminus E$, every Lipschitz map $F_0\colon E \to c_0$ has an extension $F\colon E \cup \{a\} \to c_0$ with $\operatorname{Lip}(F) \le \lambda \operatorname{Lip}(F_0)$.*

(iii) *If $a \in M \setminus E$ and $\epsilon > 0$, then there is a finite subset $\{e_1, \ldots, e_n\}$ of E so that:*

$$\min_{1 \le j \le n} d(e_j, x) \le \lambda d(a, x) + \epsilon \quad x \in E. \tag{4.1}$$

(iv) *If $a \in M \setminus E$ and $\epsilon > 0$ then for every sequence $(x_k)_{k=1}^{\infty}$ in E there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ and $e \in E$ so that*

$$d(e, x_k) \le \lambda d(a, x_k) + \epsilon \quad k \in \mathbb{M}. \tag{4.2}$$

The implications (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) $\Rightarrow$ (ii) hold without the assumption of separability of E.

Proof: (i) $\Rightarrow$ (ii) is obvious.

Let us prove (ii) $\Rightarrow$ (iii). Suppose (a, ϵ) are chosen so that (iii) fails. Let us pick any sequence $\{x_{2n-1}\}_{n=1}^{\infty}$ dense in E. Then pick a further sequence $(x_{2n})_{n=1}^{\infty}$ inductively so that

$$\lambda d(a, x_{2n}) + \epsilon < \min_{1 \le j \le 2n-1} d(x_j, x_{2n}).$$

Define $F_0\colon E \to c_0$ by

$$F_0(x) = \big(\min_{1 \le k \le n} d(x, x_k)\big)_{n=1}^{\infty}.$$

Suppose F_0 has an extension F to $E \cup \{a\}$ with $\operatorname{Lip}(F) \le \lambda \operatorname{Lip}(F_0)$. Let $F(a) = (\xi_n)_{n=1}^{\infty}$. Then for every n

$$\xi_{2n-1} \ge \min_{1 \le j \le 2n-1} d(x_{2n}, x_j) - \lambda d(a, x_{2n}) > \epsilon$$

contrary to assumption.

(iii) $\Rightarrow$ (iv) is obvious.

We next prove (iii) $\Rightarrow$ (ii). (This does not require separability of E). Suppose $F_0\colon E \to c_0$ is a Lipschitz function with $\mathrm{Lip}(F_0) = 1$. Let $F_0(x) = (\varphi_n(x))_{n=1}^{\infty}$. Then $\mathrm{Lip}(\varphi_n(x)) \le 1$. For any $a \in M$ we define

$$g_n(a) = \inf_{x \in E} (\varphi_n(x) + \lambda d(x, a))$$

and

$$h_n(a) = \sup_{x \in E} (\varphi_n(x) - \lambda d(x, a)).$$

Then $g_n(a) \ge h_n(a)$ and $\mathrm{Lip}(g_n), \mathrm{Lip}(h_n) \le \lambda$. If we define

$$f_n(a) = \begin{cases} h_n(a) & \text{if } h_n(a) \ge 0 \\ g_n(a) & \text{if } g_n(a) \le 0 \\ 0 & \text{otherwise,} \end{cases}$$

then f_n is also Lipschitz and $\mathrm{Lip}(f_n) \le \lambda$. We define $F(x) = (f_n(x))_{n=1}^{\infty}$. Clearly $F(x) = F_0(x)$ if $x \in E$ and F maps into ℓ_∞. We need only check that F has range in c_0.

Suppose $a \notin E$. If $F(a)$ is not in c_0 we can find an infinite subset $\mathbb{M}$ of $\mathbb{N}$ and $\epsilon > 0$ so that either

$$f_n(a) > 2\epsilon, \quad n \in \mathbb{M} \tag{4.3}$$

or

$$f_n(a) < -2\epsilon, \quad n \in \mathbb{M}. \tag{4.4}$$

In the former case (4.3) we have $f_n(a) = h_n(a)$ for $n \in \mathbb{M}$. Hence, there exist $(x_n)_{n\in\mathbb{M}}$ with $x_n \in E$ and

$$\varphi_n(x_n) - \lambda d(x_n, a) > 2\epsilon \quad n \in \mathbb{M}.$$

Now by (iv) we pass to a further subsequence $\mathbb{J} \subset \mathbb{M}$ so that for some $e \in E$ we have

$$d(e, x_n) < \lambda d(a, x_n) + \epsilon \quad n \in \mathbb{J}.$$

Thus

$$\varphi_n(e) > \varphi_n(x_n) - d(x_n, e) > \epsilon \quad n \in \mathbb{J}$$

which gives a contradiction since $\lim_{n\to\infty} \varphi_n(e) = 0$. The treatment of the case of (4.4) is similar. Hence $F(a) \in c_0$. ∎

THEOREM 4.2: *Let M be a metric space and suppose E is a separable subset of M. The following conditions are equivalent:*

(i) *(E, M) has the Lipschitz $(\lambda, \mathcal{C}(K))$-EP for every compact metric space K.*

(ii) *(E, M) has the Lipschitz (λ, c)-EP.*

(iii) *If K is a compact metric space and $F_0\colon E \to c$ is a Lipschitz map and $a \in M \setminus E$, then there is an extension $F\colon E \cup \{a\} \to c$ with $\operatorname{Lip}(F) \leq \lambda \operatorname{Lip}(F_0)$.*

(iv) *If $a \in M \setminus E$ and $\epsilon > 0$, there is a finite set $\{e_1, \ldots, e_n\} \subset E$ such that for any $x, y \in E$*

$$\min_{1 \leq j \leq n} (d(e_j, x) + d(e_j, y)) \leq \lambda(d(a, x) + d(a, y)) + \epsilon \quad x, y \in E. \tag{4.5}$$

(v) *If $a \in M \setminus E$ and $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are two sequences in E, there exists $e \in E$ and an infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that*

$$d(e, x_n) + d(e, y_n) \leq \lambda(d(a, x_n) + d(a, y_n)) + \epsilon \quad n \in \mathbb{N}. \tag{4.6}$$

The implications (iv) $\Rightarrow$ (v) $\Rightarrow$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) hold without the assumption of separability of E.

Proof: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear. Let us prove (iii) $\Rightarrow$ (iv) (this requires separability of E). Let $(e_n)_{n=1}^\infty$ be any dense sequence in E. If (iv) fails for some $\epsilon > 0$ we can construct sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ in E so that

$$\lambda(d(a, x_n) + d(a, y_n)) + \epsilon < d(x_j, x_n) + d(x_j, y_n) \quad j < n \tag{4.7}$$

$$\lambda(d(a, x_n) + d(a, y_n)) + \epsilon < d(y_j, x_n) + d(y_j, y_n) \quad j < n. \tag{4.8}$$

and

$$\lambda(d(a, x_n) + d(a, y_n)) + \epsilon < d(e_j, x_n) + d(e_j, y_n) \quad j < n. \tag{4.9}$$

We note that taking a subsequence, since E is separable, we can suppose that $\lim_{k \to \infty}(d(e, x_k) - d(a, x_k))$ exists for all $e \in E$. We proceed by defining a sequence of functions $f_n\colon E \to \mathbb{R}$. We first set:

$$f_{2n-1}(x) = d(x, x_n) - d(a, x_n) \quad n = 1, 2, \ldots.$$

Obviously $\operatorname{Lip}(f_{2n-1}) = 1$. We then define φ_n on the set $E_n = \{x_k\}_{k=1}^n \cup \{y_k\}_{k=1}^n \cup \{e_k\}_{k=1}^n$ by

$$\varphi_n(x_j) = f_{2n-1}(x_j), \quad \varphi_n(y_j) = f_{2n-1}(y_j) \quad j < n$$

and

$$\varphi_n(e_j) = f_{2n-1}(e_j) \quad j \leq n,$$

but

$$\varphi_n(x_n) = f_{2n-1}(x_n) + \lambda(d(x_n, a) + d(y_n, a)) - d(x_n, y_n) + \epsilon$$

and

$$\varphi_n(y_n) = f_{2n-1}(y_n) + \lambda(d(x_n, a) + d(y_n, a)) - d(x_n, y_n) + \epsilon.$$

We then claim that $\mathrm{Lip}(\varphi_n) \leq 1$. Indeed, it suffices to estimate $\varphi_n(w) - \varphi_n(z)$, when $w = x_n$ or $w = y_n$ and $z \in \{x_k\}_{k=1}^{n-1} \cup \{y_k\}_{k=1}^{n-1} \cup \{e_k\}_{k=1}^{n}$. In these case

$$\varphi_n(w) - \varphi_n(z) \geq f_{2n-1}(w) - f_{2n-1}(z) \geq -d(w, z)$$

but, using (4.7), (4.8) and (4.9), we get

$$\varphi_n(w) - \varphi_n(z) \leq d(w, x_n) + d(y_n, z) - d(x_n, y_n) \leq d(w, z)$$

by considering the cases $w = x_n$ and $w = y_n$. Now let f_{2n} be any 1-Lipschitz extension of φ_n to E.

Let $F_0(x) = (f_n(x))_{n=1}^{\infty}$. Then F_0: $E \to c$ is 1-Lipschitz. Let $F(x) = (g_n(x))_{n=1}^{\infty}$ be any extension to M with Lipschitz constant at most λ. Then

$$g_{2n}(a) \geq f_{2n}(y_n) - \lambda d(a, y_n)$$

and

$$g_{2n-1}(a) \leq f_{2n-1}(x_n) + \lambda d(a, x_n).$$

Thus

$$g_{2n}(a) - g_{2n-1}(a) \geq \epsilon$$

for every n which contradicts the fact that F maps into c.

(iv) $\Rightarrow$ (v) is immediate. Let us conclude by proving (v) $\Rightarrow$ (i). Suppose E is a nonempty subset of M and F_0: $E \to \mathcal{C}(K)$ is a Lipschitz map with $\mathrm{Lip}(F_0) \leq 1$. Let us define G, H: $M \to \ell_\infty(K)$ and H: $M \to \ell_\infty(K)$ by

$$G(x) = \inf\{F_0(e) + \lambda d(e, x) : e \in E\}$$

and

$$H(x) = \sup\{F_0(e) - \lambda d(e, x) : e \in E\}.$$

Then $\mathrm{Lip}(G), \mathrm{Lip}(H) \leq \lambda$ and $H \leq G$. We now define

$$F_u(x)(s) = \limsup_{t \to s} H(x)(t) \quad s \in K, x \in M$$

and

$$F_l(x)(s) = \liminf_{t\to s} G(x)(t) \quad x \in K, x \in M.$$

Thus $F_u(x)$ is the upper-semi-continuous regularization of $H(x)$ and $F_l(x)$ is the lower-semi-continuous regularization of $G(x)$. It is clear that $F_u\colon M \to \mathcal{U}(K)$, $F_l\colon M \to \mathcal{L}(K)$ satisfy $\mathrm{Lip}(F_u), \mathrm{Lip}(F_l) \le \lambda$ and $F_u(e) = F_l(e) = F_0(e)$ for $e \in E$.

We now need to show that $F_u(a) \le F_l(a)$ for $a \notin E$. Suppose on the contrary that $F_u(a)(s) > F_l(a)(s) + 2\epsilon$ for some $a \in M$, $s \in K$ and $\epsilon > 0$. Then there exist sequences $(s_n)_{n=1}^\infty, (s'_n)_{n=1}^\infty$ in K with $s_n, s'_n \to s$ and $H(s_n) > G(s'_n) + 2\epsilon$. Hence there exist sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ in E so that

$$F_0(y_n)(s'_n) - \lambda d(a, y_n) > F_0(x_n)(s_n) + \lambda d(a, x_n) + 2\epsilon \quad n \in \mathbb{N}.$$

Now using (v) we may find $e \in E$ and an infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that

$$d(e, x_n) + d(e, y_n) \le \lambda(d(a, x_n) + d(a, y_n)) + \epsilon \quad n \in \mathbb{M}.$$

Let $F_0(e) = f \in \mathcal{C}(K)$. Then $F_0(y_n) \le f + d(e, y_n)$ and $F_0(x_n) \ge f - d(e, x_n)$. Thus

$$f(s'_n) - f(s_n) > \lambda(d(a, y_n) + d(a, x_n)) - (d(e, x_n) + d(e, y_n)) + 2\epsilon \ge \epsilon \quad n \in \mathbb{M}.$$

This contradicts the continuity of f for large n.

Thus $F_u(a) \le F_l(a)$ and by Theorem 3.4 we can find a Lipschitz function $F\colon M \to \mathcal{C}(K)$ with $\mathrm{Lip}(F) \le \lambda$ and $F_u \le F \le F_l$ (so that F is the desired extension). ∎

In view of the equivalence of (i) and (ii) above we say that (E, M) has the **Lipschitz** $(\lambda, \mathcal{C})$**-EP** if (E, M) has the Lipschitz $(\lambda, \mathcal{C}(K))$-EP for every compact metric space K. It is clear that if (E, M) has the Lipschitz $(\lambda, \mathcal{C}(K))$-EP for some infinite compact metric space K then it has the Lipschitz (λ, c)-EP and hence the Lipschitz $(\lambda, \mathcal{C})$-EP (since c is isometric to a 1-complemented subspace of $\mathcal{C}(K)$). We thus use $\mathcal{C}$ to denote any $\mathcal{C}(K)$ for an infinite compact metric space.

Let us make a few simple deductions from these results.

PROPOSITION 4.3: *Let M be a metric space and suppose E is a subset of M. Then if (E, M) has the Lipschitz $(\lambda, \mathcal{C})$-EP, then (E, M) has the Lipschitz (λ, c_0)-EP.*

Proof: This is immediate since (4.5) reduces to (4.1) if we take $x = y$. ∎

Remark: Let us remark that (4.6) holds automatically if either:

(i) $\lambda > 1$ and one of the sequences $(x_n)_{n=1}^\infty$ or $(y_n)_{n=1}^\infty$ is metrically unbounded, or

(ii) either $(x_n)_{n=1}^\infty$ or $(y_n)_{n=1}^\infty$ has a Cauchy subsequence.

To prove the latter statement, suppose $(x_n)_{n=1}^\infty$ has a weakly Cauchy subsequence. Then there must exist an infinite subset $\mathbb{M}$ of $\mathbb{N}$ and $e \in E$ so that $d(x_n, e) < \frac{1}{2}\epsilon$ for $n \in \mathbb{M}$. Now for $n \in \mathbb{M}$ we have

$$d(x_n, e) + d(y_n, e) < d(x_n, y_n) + \epsilon \leq d(x_n, a) + d(y_n, a) + \epsilon.$$

Thus we deduce:

Proposition 4.4:

(i) *If E is a compact subset of a metric space then (E, M) has the Lipschitz $\mathcal{C}$-IEP.*

(ii) *If E has the Heine-Borel property then (E, M) has the almost isometric Lipschitz $\mathcal{C}$-EP.*

In fact, a stronger form of (i) is a consequence of a result of Espinola and Lopez ([5]), who showed that every compact subset of a $\mathcal{C}(K)$-space is contained in a compact hyperconvex subset.

Proposition 4.5: *Suppose (E, M) has the Lipschitz $(\lambda, \mathcal{C})$-EP (respectively, the Lipschitz (λ, c_0)-EP) for every $\lambda > \lambda_0$. If either $\lambda_0 > 1$ or if E is metrically bounded, then (E, M) has the Lipschitz $(\lambda_0, \mathcal{C})$-EP (respectively, the Lipschitz (λ_0, c_0)-EP).*

Proof: Assume (E, M) fails the Lipschitz $(\lambda_0, \mathcal{C})$-EP. Then there exists $a \in M \setminus E$, $\epsilon > 0$ and metrically bounded sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ in E so that (4.6) fails for $\lambda = \lambda_0$ (for any subsequence). It is then trivial to see that it fails also for some $\lambda > \lambda_0$ and $0 < \epsilon' < \epsilon$. The c_0-case is similar. ∎

Another deduction from these remarks is the following:

Proposition 4.6: *Let M be a metric space and suppose E is a subset of M. Then*

(i) *(E, M) has the Lipschitz c_0-AIEP if and only if whenever $a \notin E$, $\epsilon > 0$ and $(x_n)_{n=1}^\infty$ is a metrically bounded sequence in E there is an infinite subset $\mathbb{M}$ of $\mathbb{N}$ and $e \in E$ with*

$$d(e, x_n) < d(a, x_n) + \epsilon \quad n \in \mathbb{M}.$$

(ii) (E, M) *has the Lipschitz* $\mathcal{C}$*-AIEP if and only if whenever* $a \notin E$, $\epsilon > 0$ *and* $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty$ *are metrically bounded sequences in* E *there is an infinite subset* $\mathbb{M}$ *of* $\mathbb{N}$ *and* $e \in E$ *with*

$$d(e, x_n) + d(e, y_n) < d(a, x_n) + d(a, y_n) + \epsilon \quad n \in \mathbb{M}.$$

The consequence of this remark is that we obtain almost isometric extensions by checking bounded sequences and then the isometric extensions require additionally checking of unbounded sequences.

5. Extension properties for arbitrary metric spaces

Since both c_0 and $\mathcal{C}(K)$ for any compact metric space K are 2-absolute Lipschitz retracts it follows that any metric space has both the $(2, c_0)$-EP and $(2, \mathcal{C})$-EP.

Let us now give criteria for the Lipschitz (λ, c_0)- and $(\lambda, \mathcal{C})$-extension properties when $1 \le \lambda \le 2$. These properties can be described in terms of forbidden sequences.

THEOREM 5.1: *Let* M *be a metric space and suppose* $\lambda \ge 1$. *The following conditions on* M *are equivalent:*

(i) M *has the Lipschitz* (λ, c_0)*-EP.*

(ii) *If* $\epsilon > 0$ *and* $a \in M$, *then it is impossible to find a sequence* $(x_n)_{n=1}^\infty$ *in* M *such that*

$$\lambda d(a, x_k) + \epsilon < d(x_j, x_k) \quad 1 \le j \le k-1. \tag{5.1}$$

If $\lambda > 1$ *the sequence* $(x_k)_{k=1}^\infty$ *in* (ii) *can be assumed bounded.*

Proof: This follows from Theorem 4.1. Indeed if (ii) holds, then an easy induction argument shows that (iii) of Theorem 4.1 holds for any E and $a \in M \setminus E$ and so M has the (λ, c_0)-EP. If (4.1), fails then given $(x_1, \ldots, x_{k-1}) \in E$ (assumed to satisfy (5.1) we may find $x_k \in E$ so that (5.1) continues to hold. Conversely if (i) and there exist $(\epsilon, a, x_1, \ldots)$ so that (5.1) holds then we may take $E = \{x_k\}_{k=1}^\infty$ and now (iv) of Theorem 4.1 fails. Since E is countable this implies that (E, M) fails the Lipschitz (λ, c_0)-EP contrary to assumption. ∎

THEOREM 5.2: *Let* (M, d) *be a metric space and suppose* $\lambda \ge 1$. *Then the following conditions on* M *are equivalent:*

(i) M *has the Lipschitz* $(\lambda, \mathcal{C})$*-EP.*

(ii) *If $\epsilon > 0$ and $a \in M$ then it is impossible to find sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in M such that*

$$\lambda(d(a,x_k)+d(a,y_k))+\epsilon < d(x_j,x_k)+d(x_j,y_k) \quad 1 \le j \le k-1 \tag{5.2}$$

and

$$\lambda(d(a,x_k)+d(a,y_k))+\epsilon < d(y_j,x_k)+d(y_j,y_k) \quad 1 \le j \le k-1. \tag{5.3}$$

If $\lambda > 1$ the sequences $(x_k)_{k=1}^{\infty}$ and $(y_k)_{k=1}^{\infty}$ may be assumed metrically bounded.

Proof: This follows in a similar way from Theorem 4.2. Indeed if (ii) holds then we can verify (iv) of Theorem 4.2 for any subset E and $a \in M \setminus E$ by a simple induction argument. As before, if (4.5) does not hold then given $\{x_1, \ldots, x_{k-1}, y_1, \ldots, y_{k-1}\} \subset E$ satisfying (5.2) and (5.3), we may find x_k, y_k so that both (5.2) and (5.3) continue to hold.

Conversely if (i) then if (5.2) and (5.3) hold, taking $E = \{x_k\}_{k=1}^{\infty} \cup \{y_k\}_{k=1}^{\infty}$ we contradict (v) of Theorem 4.2. ∎

Remarks: It should be observed here that M has the Lipschitz $\mathcal{C}$-IEP if and only if we cannot find $\epsilon > 0$, $a \in M$ and sequences $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}$ in M satisfying (5.2) and (5.3) for $\lambda = 1$. On the other hand M has the Lipschitz $\mathcal{C}$-AIEP if and only if we cannot find $\epsilon > 0$, $a \in M$ and metrically bounded sequences $(x_n)_{n=1}^{\infty}, (y_n)_{n=1}^{\infty}$ in M satisfying (5.2) and (5.3) for $\lambda = 1$. This follows from Proposition 4.6.

We also note that (5.2) and (5.3) cannot hold if either of sequences $(x_n)_{n=1}$ or $(y_n)_{n=1}^{\infty}$ has a Cauchy subsequence. Indeed suppose we can find $n > k$ so that $d(x_n, x_k) < \frac{1}{2}\epsilon$. Then

$$d(x_n,x_k)+d(y_n,x_k) < d(y_n,x_n)+\epsilon \le d(a,x_n)+d(a,y_n)+\epsilon.$$

(See the Remark after Proposition 4.3.)

It will also be useful to note the following:

Theorem 5.3: *Let (M,d) be a metric space and suppose $\lambda \ge 1$. Then the following conditions on M are equivalent:*

(i) *M has the Lipschitz $(\lambda, \mathcal{C})$-EP and the Lipschitz $(\lambda, \mathcal{C})$-UEP.*

(ii) *Every subset of M has the Lipschitz $(\lambda, \mathcal{C})$-EP.*

(iii) *Whenever M is isometrically embedded in some metric space M', $\epsilon > 0$ and $a \in M'$ then it is impossible to find sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ in M such that*

$$\lambda(d(a,x_k)+d(a,y_k))+\epsilon < d(x_j,x_k)+d(x_j,y_k) \quad 1 \le j \le k-1 \tag{5.4}$$

and

$$\lambda(d(a,x_k)+d(a,y_k))+\epsilon < d(y_j,x_k)+d(y_j,y_k) \quad 1 \le j \le k-1. \tag{5.5}$$

If $\lambda > 1$ the sequences $(x_k)_{k=1}^{\infty}$ and $(y_k)_{k=1}^{\infty}$ may be assumed metrically bounded.

Proof: (i)$\Leftrightarrow$(ii) is Proposition 2.1.

(ii)$\Rightarrow$(iii). Again, this follows by using Theorem 4.2 for the case

$$E = \{x_k\}_{k=1}^{\infty} \cup \{y_k\}_{k=1}^{\infty}.$$

(iii)$\Rightarrow$(i) is similar to the previous theorem. ∎

As in the preceding case it is impossible to have (5.4) and (5.5) when either sequence $(x_n)_{n=1}^{\infty}$ or $(y_n)_{n=1}^{\infty}$ has a Cauchy subsequence.

Corollary 5.4: *(i) If M has the Lipschitz $(\lambda,\mathcal{C})$-EP then M has the Lipschitz (λ,c_0)-EP.*
(ii) If M has the Lipschitz (λ,c_0)-EP then M has the Lipschitz $(1+\frac{1}{2}\lambda),\mathcal{C})$-EP.

In particular M fails the Lipschitz $(\lambda,\mathcal{C})$-EP for every $\lambda < 2$ if and only if M fails the Lipschitz (λ,c_0)-EP for every $\lambda < 2$.

Proof: (i) follows from Proposition 4.3. For (ii) we suppose $\mu = 1+\frac{1}{2}\lambda$ and that $a \in M$, $\epsilon > 0$ and $(x_k)_{k=1}^{\infty}, (y_k)_{k=1}^{\infty}$ are bounded sequences chosen so that

$$\mu(d(a,x_k)+d(a,y_k))+\epsilon < d(x_j,x_k)+d(x_j,y_k) \quad 1 \le j \le k-1 \tag{5.6}$$

and

$$\mu(d(a,x_k)+d(a,y_k))+\epsilon < d(y_j,x_k)+d(y_j,y_k) \quad 1 \le j \le k-1. \tag{5.7}$$

We may assume without loss of generality, by passing to a subsequence, that the limits $\lim_{n\to\infty} d(a,x_n) = \xi$ and $\lim_{n\to\infty} d(a,y_n) = \eta$ exist and that $\xi \le \eta$. Now by Theorem 5.1 (iv) given any m_0 there exists $m \ge m_0$ and an infinite subset $\mathbb{M}$ of $\mathbb{N}$ such that

$$d(x_m,x_n) \le \lambda d(x_n,a) + \frac{1}{2}\epsilon \quad n \in \mathbb{M}.$$

and

$$d(x_m, y_n) \leq d(x_m, a) + d(y_n, a).$$

Hence for $n \in \mathbb{M}$ with $n > m$,

$$\mu(d(a, x_n) + d(a, y_n)) + \frac{1}{2}\epsilon < \lambda d(a, x_n) + d(x_m, a) + d(y_n, a).$$

Letting $n \to \infty$ we have

$$(1 + \frac{1}{2}\lambda)(\xi + \eta) + \frac{1}{2}\epsilon \leq \lambda\xi + \eta + d(x_m, a).$$

Now we can let $m \to \infty$ and obtain

$$(1 + \frac{1}{2}\lambda)(\xi + \eta) + \frac{1}{2}\epsilon \leq \lambda\xi + \eta + \xi$$

i.e.

$$\lambda(\eta - \xi) + \frac{1}{2}\epsilon \leq 0$$

contrary to assumption. ∎

THEOREM 5.5: *Suppose X is a finite-dimensional normed space. Then every subset of X has the Lipschitz c_0-UIEP property; in particular, X has the Lipschitz c_0-IEP.*

Proof: Suppose $E \subset X$ is embedded in a metric space M and $a \in M \setminus E$. We use (iv) of Theorem 4.1. Fix $\epsilon > 0$. We consider a sequence in E, $(x_n)_{n=1}^\infty$. If $(x_n)_{n=1}^\infty$ has a bounded subsequence then it follows from the Heine–Borel property of X that (4.2) is satisfied for some $\mathbb{M}$. We therefore may assume passing to a subsequence $\|x_1\| > 0$ and that $\|x_n\| > n\|x_{n-1}\|$ for $n \geq 2$ and that $y_n = x_n/\|x_n\|$ is convergent to some $y \in Y$. Assume that for every k the set of n such that

$$\|x_n - x_k\| < d(a, x_n) + \epsilon$$

is finite. Then by passing to a subsequence we can suppose that

$$\|x_n - x_k\| \geq d(a, x_n) + \epsilon \quad n > k.$$

Let $x_{n,k}^*$ be a norming functional for $x_n - x_k$ i.e., $\|x_{n,k}^*\| = 1$ and $x_{n,k}^*(x_n - x_k) = \|x_n - x_k\|$. Then

$$x_{n,k}^*(x_n - x_k) \geq \|x_n\| - d(a, 0) + \epsilon \geq x_{n,k}^*(x_n) - d(a, 0) + \epsilon.$$

Hence,

$$x_{n,k}^*(x_k) \leq d(a, 0) - \epsilon \quad n > k.$$

Let y_k^* be a cluster point of $(x_{n,k}^*)_{n=1}^\infty$. Then $y_k^*(y) = 1$ but

$$y_k^*(y_k) \leq \|x_k\|^{-1}(d(a,0) - \epsilon) \quad k \in \mathbb{N}.$$

If y^* is a cluster point of $(y_k^*)_{k=1}^\infty$, then it follows that $y^*(y) = 0$ which gives a contradiction. ∎

THEOREM 5.6: *Let M be a metric space. The following conditions on M are equivalent:*

(i) *M has the Heine–Borel property.*
(ii) *Every subset of M has the Lipschitz $\mathcal{C}$-UAIEP.*
(iii) *Every subset of M has the Lipschitz $(\lambda, \mathcal{C})$-UEP for some $\lambda < 2$.*
(v) *Every subset of M has the Lipschitz (λ, c_0)-UEP for some $\lambda < 2$.*

Proof: (i) ⇒ (ii) follows from Proposition 4.5.

(ii) ⇒ (iii) is trivial.

(iii) ⇒ (iv) follows from Proposition 4.3.

(iv) ⇒ (i) If M fails the Heine–Borel property for any $\mu > 1$ we can find an infinite sequence $(x_n)_{n=1}^\infty$ such that for some $\delta > 0$ we have

$$\delta < d(x_j, x_k) < \mu\delta \quad j,k \in \mathbb{N}.$$

Let $E = \{x_j\}_{j=1}^\infty$ and adjoin a point a to M such that $d(a, x_j) = \frac{1}{2}\mu\delta$ for all $j \in \mathbb{N}$. Let $\epsilon = \nu\delta$ where $\nu > 0$. We see that the hypotheses of Theorem 4.1 (v) can only hold if

$$1 \leq \frac{1}{2}\lambda\mu + \nu.$$

Since $\mu > 1$ and $\nu > 0$ are arbitrary this reaches a contradiction if $\lambda < 2$. ∎

Example: The following example shows that we cannot prove a similar result to Theorem 5.6 for the isometric case. Consider the metric on $\mathbb{N}$ defined by $d(m,n) = \max(m,n)$ for $m \neq n$. It is easy to see that the space $(\mathbb{N}, d)$ has the Heine–Borel property and the Lipschitz $\mathcal{C}$-IEP. However if one adjoins 0 and defines $d(0,n) = n - 1/2$ then it is clear in Theorem 4.1 that (iv) fails for $a = 0$, $x_n = n$ and $\epsilon < 1/2$. Thus this space fails to have the Lipschitz $\mathcal{C}$-UIEP or Lipschitz c_0-UIEP.

Let us note at this point that it is easy to give examples of metric spaces which fail the Heine–Borel property but nevertheless have the $\mathcal{C}$-IEP. This is a consequence of the following

PROPOSITION 5.7: *Let (M, d) be an ultrametric space; then M has the Lipschitz $\mathcal{C}$-IEP.*

Proof: Suppose we can find $a \in M$ $\epsilon > 0$ and two sequences $(x_n)_{n=1}, (y_n)_{n=1}^{\infty}$ such that (5.2) and (5.3) both hold. Let

$$\sigma_k = \min(d(x_k, a), d(y_k, a)) \quad k = 1, 2, \dots$$

Fix k and suppose for example that $d(x_k, a) = \sigma_k$. Then

$$d(x_{k+1}, x_k) \leq \max(\sigma_k, d(x_{k+1}, a)), d(y_{k+1}, x_k) \leq \max(\sigma_k, d(y_{k+1}, a)).$$

Hence either

$$d(x_{k+1}, a) < \sigma_k - 1/2\epsilon$$

or

$$d(y_{k+1}, a) < \sigma_k - 1/2\epsilon.$$

This implies

$$\sigma_{k+1} < \sigma_k - 1/2\epsilon \quad k = 1, 2, \dots$$

which gives a contradiction. ∎

If X is an infinite-dimensional Banach space then the **Kottman constant** [7] of X is defined by:

$$\kappa(X) = \sup_{x_n \in B_X} \operatorname{sep}(x_n)$$

where for any sequence $(x_n)_{n=1}^{\infty}$ we define

$$\operatorname{sep}(x_n) = \inf_{m \neq n} \|x_m - x_n\|.$$

A result of Elton and Odell [4] asserts that $\kappa(X) > 1$ for every infinite-dimensional Banach space. See also [8] for a recent lower estimate for $\kappa(X)$ for X non-reflexive.

It is an immediate consequence of Theorem 5.1 that:

PROPOSITION 5.8: *If X is an infinite-dimensional Banach space, then X has the Lipschitz (λ, c_0)-IEP if and only if $\lambda \geq \kappa(X)$.*

In particular, X fails the Lipschitz c_0-AIEP: this result is due to Lancien and Randrianantoanina [9].

We remark that if $1 \leq p < \infty$ then $\kappa(\ell_p) = 2^{1/p}$ while $\kappa(c_0) = 2$ ([7] and [3]). We next show that for these spaces if $1 \leq p \leq 2$, the Lipschitz (λ, c_0)-EP and the Lipschitz $(\lambda, \mathcal{C})$-EP are equivalent:

PROPOSITION 5.9: *For $1 \le p \le 2$, ℓ_p has the Lipschitz $(\lambda, \mathcal{C})$-EP if and only if $\lambda \ge \kappa(\ell_p) = 2^{1/p}$. Similarly c_0 has the Lipschitz $(\lambda, \mathcal{C})$-EP if and only if $\lambda = 2$.*

Proof: For the case of ℓ_1 and c_0 this is immediate from Corollary 5.4. We consider the case $1 < p \le 2$. We will need the following inequality:

$$2^{1/p}s + (s^p + t^p + (u+v)^p)^{1/p} \le 2^{1/p}((s^p + u^p)^{1/p} + (t^p + v^p)^{1/p}) \qquad 0 \le s \le t,\ u, v \ge 0. \tag{5.8}$$

We first note that to establish (5.8) we need only consider the case $s = t$. This follows since

$$2^{1/p}(t^p + v^p)^{1/p} - (s^p + t^p + (u+v)^p)^{1/p}$$

is increasing as a function of t if s, u, v are fixed. Thus we need to prove:

$$s + (s^p + 1/2(u+v)^p)^{1/p} \le (s^p + u^p)^{1/p} + (s^p + v^p)^{1/p} \quad 0 \le s, u, v. \tag{5.9}$$

In fact, using the concavity of $t \to t^{1/p}$, the fact that $p \le 2$ and the triangle law in ℓ_p we have

$$\begin{aligned} s + (s^p + 1/2(u+v)^p)^{1/p} &\le 2(s^p + 1/4(u+v)^p)^{1/p} \\ &\le 2(s^p + (1/2(u+v))^p)^{1/p} \\ &\le (s^p + u^p)^{1/p} + (s^p + v^p)^{1/p} \end{aligned}$$

and this proves (5.9) and hence (5.8).

Now suppose $a \in \ell_p$ and $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty$ are two bounded sequences. We will verify (ii) of Theorem 5.2 with $\lambda = 2^{1/p}$. Assume that $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty, a, \epsilon$ satisfy (5.2) and (5.3) and that both $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are bounded sequences. We may assume that $\lim_{n\to\infty} \|x_n - a\|$ and $\lim_{n\to\infty} \|y_n - a\|$ both exist and that $\lim_{n\to\infty} x_n = x$ weakly and $\lim_{n\to\infty} y_n = y$ weakly. Further we can suppose that $\lim_{n\to\infty} \|u_n\| = \xi$ and $\lim_{n\to\infty} \|v_n\| = \eta$ exist where $u_n = x_n - x$ and $v_n = y_n - y$. Let us suppose $\xi \le \eta$.

Then if $m > n$,

$$2^{1/p}(\|x_m - a\| + \|y_m - a\|) + \epsilon < \|x_m - x_n\| + \|y_m - x_n\|.$$

We shall show, however, that

$$2^{1/p} \lim_{m\to\infty} (\|x_m - a\| + \|y_m - a\|) \ge \lim_{n\to\infty} \lim_{m\to\infty} (\|x_m - x_n\| + \|y_m - x_n\|) \tag{5.10}$$

(and all limits exist) and this gives us a contradiction.

Now,

$$\lim_{m\to\infty}\|x_n - x_m\| = \lim_{m\to\infty}\|u_m - u_n\| = (\|u_n\|^p + \xi^p)^{1/p}.$$

Thus

$$\lim_{n\to\infty}\lim_{m\to\infty}\|x_n - x_m\| = 2^{1/p}\xi.$$

Similarly

$$\lim_{m\to\infty}\|y_m - x_n\| = (\|y - x_n\|^p + \eta^p)^{1/p}$$

so that

$$\lim_{n\to\infty}\lim_{m\to\infty}\|y_m - x_n\|^p = (\|y - x\|^p + \eta^p + \xi^p)^{1/p}.$$

Now, using (5.8)

$$\begin{aligned}
2^{1/p}\xi &+ (\|y - x\|^p + \eta^p + \xi^p)^{1/p}\\
&\le 2^{1/p}\xi + ((\|a - y\| + \|a - x\|)^p + \eta^p + \xi^p)^{1/p}\\
&\le 2^{1/p}\Big((\|a - x\|^p + \xi^p)^{1/p} + (\|a - y\|^p + \eta^p)^{1/p}\Big)\\
&= 2^{1/p}(\lim_{n\to\infty}\|x_n - a\| + \lim_{n\to\infty}\|y_n - a\|).
\end{aligned}$$

This proves (5.10) and hence the Proposition. ∎

Note the argument of Proposition 5.9 fails for $2 < p < \infty$. We are very grateful to Yves Dutrieux for pointing out an error in an earlier version of this Proposition.

Proposition 5.10: *$c_{0,+}$ has the Lipschitz c_0-IEP.*

Proof: Suppose a, $(x_n)_{n=1}^\infty$ $\epsilon > 0$ satisfy (4.2) for $\lambda = 1$. Let P_k: $c_{0,+} \to c_{0,+}$ be the map $P_k(\xi) = (\xi_1, \ldots, \xi_k, 0, \ldots)$ and let $Q_k = I - P_k$. Fix n so that $\|Q_n a\| < \epsilon/2$. We now argue by Ramsey's theorem that there is an infinite subset $\mathbb{A}$ of $\mathbb{N}$ so that either

$$\|x_k - x_j\| = \|P_n(x_k - x_j)\| \quad j, k \in \mathbb{A}$$

or

$$\|x_k - x_j\| = \|Q_n(x_k - x_j)\| \quad j, k \in \mathbb{A}.$$

Let us consider the first case. Then the sequence $(P_n x_k)_{k\in\mathbb{A}}$ and $P_n a$ satisfy (5.1) and thus contradict Theorem 5.5 for ℓ_∞^n. Hence the second case must hold.

Now it easy to see that

$$\|Q_n(x_k - x_j)\| \le \max\{\|Q_n x_k\|, \|Q_n x_j\|\}.$$

On the other hand

$$\|x_k - a\| \geq \|Q_n x_k\| - \frac{1}{2}\epsilon.$$

Hence if $k, j \in \mathbb{A}$ with $k > j$ we have

$$\|Q_n x_j\| > \|x_k - a\| + \epsilon > \|Q_n x_k\| + \frac{1}{2}\epsilon.$$

This is clearly impossible since $\mathbb{A}$ is infinite. ∎

Remark: Note that $c_{0,+}$ cannot have the Lipschitz c_0-UAIEP because it fails the Heine–Borel property. See Theorem 5.6 above.

PROPOSITION 5.11: *$c_{0,+}$ has the Lipschitz $(\lambda, \mathcal{C})$-EP if and only if $\lambda \geq 3/2$.*

Proof: By Corollary 5.4 it is clear that $c_{0,+}$ has the $(\lambda, \mathcal{C})$-EP if $\lambda \geq 3/2$. For the other direction define $a = e_1$ and two sequences $x_n = 2e_1 + e_{n+1}$ and $y_n = e_{n+1}$. Then

$$d(x_n, a) + d(y_n, a) = 2 \quad n = 1, 2, \ldots$$

and

$$d(x_n, x_k) + d(y_n, x_k) = d(x_n, y_k) + d(y_n, y_k) = 3 \quad k < n.$$

Applying Theorem 5.2 gives the result. ∎

6. Collinearity properties

Suppose (M, d) is a metric space. We will say that the ordered triple of points $\{x_1, x_2, x_3\}$ is **metrically collinear** if

$$d(x_1, x_3) = d(x_1, x_2) + d(x_2, x_3).$$

We say that $\{x_1, x_2, x_3\}$ are **ϵ-collinear** where $\epsilon > 0$ if

$$d(x_1, x_3) > d(x_1, x_2) + d(x_2, x_3) - \epsilon. \tag{6.1}$$

Note that if $\{x_1, x_2, x_3\}$ is ϵ-collinear then so is $\{x_3, x_2, x_1\}$ but it may or may not be the case that $\{x_1, x_3, x_2\}$ is ϵ-collinear.

Let us say that a metric space (M, d) has **the collinearity property**, if, for every infinite subset $A \subset M$ and every $\epsilon > 0$, there are three distinct points $x_1, x_2, x_3 \in A$ such that $\{x_1, x_2, x_3\}$ are ϵ-collinear.

This concept appears to be new, but arose independently at the same time in the work of Melleray [12]. Melleray characterizes metric spaces M with the collinearity property as those for which the space $E(X)$ of Katetov maps is separable; we refer to his work for further information.

Proposition 6.1: *Let (M,d) be a metric space. In order that M has the collinearity property it is necessary and sufficient that:*

(a) *M has the Heine–Borel property, and*
(b) *Whenever $(x_n)_{n=1}^{\infty}$ is an unbounded sequence in M and $\epsilon > 0$ there is a subsequence $(x_{n\in\mathbb{A}})$ so that for every $j,k,n \in \mathbb{A}$ with $j<k<n$ the triple $\{x_j,x_k,x_n\}$ is ϵ-collinear.*

Proof: Suppose first that M has the collinearity property.

Suppose $(x_n)_{n=1}^{\infty}$ is a bounded sequence in M such that $\inf_{m\neq n} d(x_m,x_n) > 0$. By standard Ramsey theory we can find an infinite subset $\mathbb{A}$ of $\mathbb{N}$ such that for some constant c we have $c \le d(x_m,x_n) \le 3c/2$ for $m \neq n$ and $m,n \in \mathbb{A}$. But then $(x_n)_{n\in\mathbb{A}}$ violates (6.1) for $\epsilon = c$. This proves (a).

For (b) we observe that we may pass to a subsequence $\mathbb{J}$ so that $d(x_j,x_k) > 3\epsilon$ if $j,k \in \mathbb{J}$ and

$$d(x_n,x_j) > 2d(x_k,x_j) \quad j,k<n,\ j,k,n \in \mathbb{J}_1.$$

Now by Ramsey's theorem there is a further infinite subset $\mathbb{J}_1$ so that if $j<k<n$ with $j,k,n \in \mathbb{J}_1$ then either $\{x_j,x_k,x_n\}$ or $\{x_k,x_j,x_n\}$ is ϵ-collinear. Here we use the fact that (x_j,x_k) must be the shortest side in the triangle.

Finally applying Ramsey's theorem once more, we find an infinite subset $\mathbb{A}$ of $\mathbb{J}_1$ so that either we have:

$$d(x_n,x_j) > d(x_n,x_k) + d(x_k,x_j) - \epsilon \quad j<k<n,\ j,k,n \in \mathbb{A}$$

or we have

$$d(x_n,x_k) > d(x_n,x_j) + d(x_k,x_j) - \epsilon \quad j<k<n,\ j,k,n \in \mathbb{A}.$$

Let us prove that the second alternative is impossible. Indeed this implies that

$$d(x_n,x_j) < d(x_n,x_k) - 2\epsilon \quad j<k<n,\ j,k,n \in \mathbb{A}.$$

Now suppose $j<k<l<n$ with $j,k,l,n \in \mathbb{A}$. Then

$$d(x_n,x_l) \le d(x_n,x_j) + d(x_j,x_l) < d(x_n,x_k) + d(x_k,x_l) - 2\epsilon.$$

This is a contradiction and establishes (b).

Conversely if (a) and (b) hold, suppose $(x_n)_{n=1}^{\infty}$ is a sequence in M. Then if $(x_n)_{n=1}^{\infty}$ is unbounded we may use (b) to find $j<k<n$ so that

$$d(x_n,x_j) > d(x_n,x_k) + d(x_k,x_j) - \epsilon.$$

If $(x_n)_{n=1}^\infty$ is bounded it has an accumulation point x in the completion of M. Pick x_j, x_k, x_l so that $d(x_j,x), d(x_k,x), d(x_l,x) < \frac{1}{2}\epsilon$, and it is clear that (6.1) is satisfied. ∎

PROPOSITION 6.2: *Suppose $(M_i)_{i=1}^n$ are metric spaces with the collinearity property. Then both $(\sum_{i=1}^n M_i)_{\ell_1}$ and $(\sum_{i=1}^n M_i)_{\ell_\infty}$ have the collinearity property.*

Proof: In both spaces it is clear that we have the Heine–Borel property.

Now suppose $(x_k)_{n=1}^\infty$ is an unbounded sequence in $(\sum_{j=1}^n M_j)_{\ell_1}$. Letting $x_k = (\xi_{k1}, \ldots, \xi_{kn})$ by repeated use of Proposition 6.1 given $\epsilon > 0$ there is an infinite subset $\mathbb{A}$ of $\mathbb{N}$ so that for each $1 \le i \le n$ we have either

$$d_i(\xi_{li}, \xi_{ki}) + d_i(\xi_{ki}, \xi_{ji}) < d_i(\xi_{li}, \xi_{ji}) + \epsilon/n \quad l > k > j,\ l,k,j \in \mathbb{A}$$

or

$$d_i(\xi_{li}, \xi_{ki}) < \epsilon/2n \quad l > k,\ l,k \in \mathbb{A}.$$

It then follows that

$$d(x_l, x_k) + d(x_k, x_j) < d(x_l, x_j) + \epsilon \quad l > k > j,\ l,k,j \in \mathbb{A}$$

and hence $(\sum_{j=1}^n M_j)_{\ell_1}$ has the collinearity property.

Next suppose $(x_k)_{n=1}^\infty$ is an unbounded sequence in $(\sum_{j=1}^n M_j)_{\ell_\infty}$. Then by Ramsey's theorem there is an infinite subset $\mathbb{A}$ of $\mathbb{N}$ and a fixed $1 < i < n$ so that

$$d(x_k, x_j) = d_i(\xi_{ki}, \xi_{ji}) \quad k > j,\ k,j \in \mathbb{A}.$$

The conclusion follows using the collinearity property in M_i. ∎

LEMMA 6.3: *Let E be a subset of a metric space M and $a \in M$. Suppose E has the collinearity property. Then, for any sequence $(x_n)_{n=1}^\infty$ in E and any $\epsilon > 0$ there is an infinite subset $(x_n)_{n\in\mathbb{A}}$ such that $\{a, x_k, x_n\}$ is ϵ-collinear whenever $k < n$ with $k, n \in \mathbb{A}$.*

Proof: If $(x_n)_{n=1}^\infty$ is bounded then since E has the collinearity property we can pass to a subsequence $(x_n)_{n\in\mathbb{A}}$ such that $d(x_k, x_n) < \frac{1}{2}\epsilon$ for $k, n \in \mathbb{A}$ and it is then trivial that $\{a, x_k, x_n\}$ is ϵ-collinear.

If $(x_n)_{n=1}^\infty$ we can pass by Proposition 6.1 to a subsequence $(x_n)_{n\in\mathbb{A}}$ so that $\{x_j, x_k, x_n\}$ is $\frac{1}{2}\epsilon$-collinear when $j, k, n \in \mathbb{A}$ and $j < k < n$ and also such that the limit

$$\lim_{n\to\infty} d(x_n, a) - d(x_n, x_1) = \xi$$

exists. We can further require that

$$|d(x_n, a) - d(x_n, x_1) - \xi| < \frac{1}{4}\epsilon \quad 1 < n \in \mathbb{A}.$$

Then if $1 < k < n$ with $k, n \in \mathbb{A}$ we have

$$d(x_n, a) - d(x_k, a) > d(x_n, x_1) - d(x_k, x_1) - \frac{1}{2}\epsilon > d(x_n, x_k) - \epsilon$$

and we are done. ∎

We recall that a finite-dimensional normed space X is **polyhedral** if the unit ball B_X is a polyhedron, or, equivalently X isometrically embeds into ℓ_∞^m for some m. The next theorem is a close relative of Theorem 7.7 in [11].

Theorem 6.4: *A finite-dimensional normed space X is polyhedral if and only if X has the collinearity property.*

Proof: First note that ℓ_∞^n has the collinearity property by Proposition 6.2. Thus any polyhedral space has the collinearity property.

For converse assume X is finite-dimensional but not polyhedral. Then X contains a two dimensional subspace X_0 which is also not polyhedral [6]. Thus there is a point $u \in X_0$ with $\|u\| = 1$ so that u is an accumulation point of extreme points in B_{X_0}. It is clear that we can choose $v \in X_0$ so that if $F(t) = \|u + tv\| - 1$ then F has a minimum at $t = 0$, the right-hand derivative satisfies $F'_+(t) = 0$ and also $F(t) > 0$ whenever $t > 0$. Observe that $\lim_{t\to 0+} F'_-(t) = F'_+(0) = 0$.

We will now select a sequence $x_n = s_n(u + t_n v)$ with $s_n, t_n > 0$ so that $\|x_m - x_n\| < \|x_m\| - \|x_n\| - 1$ if $m > n$. This will complete the proof by taking $a = 0$ in the preceding Lemma.

We will select $(s_n, t_n)_{n=1}^\infty$ inductively so that $\|x_m - x_n\| < \|x_m\| - \|x_n\| - 1$ if $m > n$ and additionally $\|x_n\| - s_n = s_n F(t_n) > 1$ for all n. Pick $s_1, t_1 > 0$ arbitrarily so that $s_1 F(t_1) > 1$. Now suppose $(x_k)_{k<n}$ have been selected. To pick (s_n, t_n) we observe that

$$\lim_{t\to 0+} \lim_{s\to\infty} sF(t) = \infty$$

and for each fixed $k < n$,

$$\begin{aligned}
&\lim_{t\to 0+}\lim_{s\to\infty} \|s(u+tv)\| - \|s(u+tv) - s_k(u+t_kv)\| \\
&\qquad = \lim_{t\to 0+}\lim_{s\to\infty} s(1+F(t)) - (s-s_k)\Big(1+F\Big(\frac{st-s_kt_k}{s-s_k}\Big)\Big) \\
&\qquad = \lim_{t\to 0+}\lim_{s\to\infty} s_k + sF(t) - (s-s_k)F\Big(t - \frac{s_k}{s-s_k}(t_k-t)\Big) \\
&\qquad = \lim_{t\to 0+}\lim_{s\to\infty} s_k + s_kF(t) + (s-s_k)\Big(F(t) - F\Big(t-\frac{s_k}{s-s_k}(t_k-t)\Big)\Big) \\
&\qquad = \lim_{t\to 0+} s_k + s_kF(t) + s_k(t_k-t)F'_-(t) \\
&\qquad = s_k \\
&\qquad < \|x_k\| - 1.
\end{aligned}$$

Hence it is possible to choose s_n, t_n so that $x_n = s_n(u+t_nv)$ verifies the inductive hypothesis and the proof is complete. ∎

THEOREM 6.5: *Let M be a metric space with the collinearity property. Then M has the Lipschitz $\mathcal{C}$-UIEP and the Lipschitz $\mathcal{C}$-IEP.*

Proof: Suppose M is embedded in a metric space (M', d). Suppose $a \in M' \setminus M$. We verify (v) of Theorem 4.2 (i.e. condition (4.6)). Suppose $\epsilon > 0$ and $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty$ are two sequences in M. We can select an infinite subset $\mathbb{M}$ of $\mathbb{N}$ so that if $k, n \in \mathbb{M}$ and $k < n$ then $\{a, x_k, x_n\}$ are ϵ-collinear. Now for fixed $k \in \mathbb{M}$ and $n \in \mathbb{M}$ with $n \in \mathbb{M}$,

$$\begin{aligned}
d(x_k, x_n) + d(x_k, y_n) &< d(a, x_n) - d(a, x_k) + \epsilon + d(x_k, y_n) \\
&\le d(a, x_n) + d(a, y_n) + \epsilon.
\end{aligned}$$

This implies the fact that M has the Lipschitz $\mathcal{C}$-UIEP.

Now any subset of M also has the collinearity property and so also has the Lipschitz $\mathcal{C}$-UIEP. Hence M has the Lipschitz $\mathcal{C}$-IEP. ∎

The following Theorem extends the result of Lancien and Randrianantoanina [9] who proved that every finite-dimensional polyhedral space has the Lipschitz $\mathcal{C}$-IEP. Let us remark that the linear version of this result is due to Lindenstrauss [11]: a finite-dimensional normed space X has the Lipschitz $\mathcal{C}$-UILEP if and only if X is polyhedral.

THEOREM 6.6: *Let X be a finite-dimensional normed space. Then X has the Lipschitz $\mathcal{C}$-UIEP if and only if X is polyhedral.*

In particular, if X is polyhedral then every subset of X has the Lipschitz $\mathcal{C}$-UIEP and X has the Lipschitz $\mathcal{C}$-IEP.

Proof: If X is polyhedral then Theorem 6.5 and Theorem 6.4 give the conclusion. Conversely if X has the Lipschitz $\mathcal{C}$-UIEP, then (X, Y) has the $\mathcal{C}$-ILEP for every normed space $Y \supset X$ with $\dim Y/X = 1$ by Lemma 2.2. By the Corollary to Theorem 7.5 of [11] we obtain that X is polyhedral. ∎

The collinearity property is not necessary for a metric space M to have the Lipschitz $\mathcal{C}$-UIEP (since ℓ_∞ has the Lipschitz $\mathcal{C}$-UIEP). It is also not necessary even if we require that every subset of M has the Lipschitz $\mathcal{C}$-UIEP. This follows from the following theorem.

Theorem 6.7: *Let X be a strictly convex finite-dimensional normed space. Let K be a closed subset of X such that $\lambda K \subset K$ for every $\lambda \geq 0$ but $K \cap (-K) = \emptyset$ (e.g. K is a proper closed cone). Then every subset of K has the Lipschitz $\mathcal{C}$-UIEP.*

Proof: We use Theorem 5.3. Suppose K is embedded in a metric space M and $a \in M \setminus K$. Assume that we have a pair of sequences $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty$ in K such that

$$d(a, x_n) + d(a, y_n) + \epsilon < \|x_n - x_k\| + \|y_n - x_k\| \quad 1 \leq k \leq n-1$$

and

$$d(a, x_n) + d(a, y_n) + \epsilon < \|x_n - y_k\| + \|y_n - y_k\| \quad 1 \leq k \leq n-1.$$

Since K has the Heine–Borel property the sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ cannot have bounded subsequences. Therefore we have $\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = \infty$. By passing to a subsequence we can suppose that both $(x_n/\|x_n\|)$ and $(y_n/\|y_n\|)$ are convergent to u, v respectively. We have $\|u\| = \|v\| = 1$ and $u, v \in K$. Now for $1 \leq k \leq n-1$, let $x_{n,k}^* \in X^*$ be chosen so that $\|x_{n,k}^*\| = 1$ and $x_{n,k}^*(x_n - x_k) = \|x_n - x_k\|$. Similarly chose $y_{n,k}^*$ with $\|y_{n,k}^*\| = 1$ and $y_{n,k}^*(y_n - x_k) = \|y_n - x_k\|$. Thus

$$d(a, x_n) + d(a, y_n) + \epsilon < x_{n,k}^*(x_n - x_k) + y_{n,k}^*(y_n - x_k) \quad 1 \leq k \leq n-1.$$

Hence

$$\begin{aligned} x_{n,k}^*(x_n) + y_{n,k}^*(y_n) &\leq \|x_n\| + \|y_n\| \leq d(a, x_n) + d(a, y_n) + 2d(a, 0) \\ &< x_{n,k}^*(x_n - x_k) + y_{n,k}^*(y_n - x_k) + 2d(a, 0), \end{aligned}$$

so that

$$x^*_{n,k}(x_k) + y^*_{n,k}(x_k) \le 2d(a,0) \quad k < n.$$

Now for fixed k let u^*_k, v^*_k be accumulation points of the sequences $(x^*_{n,k})_{n>k}$ and $(y^*_{n,k})_{n>k}$. Then $u^*_k(u) = 1$ and $v^*_k(v) = 1$ and

$$(u^*_k + v^*_k)(x_k) \le 2d(a,0) \quad k = 1, 2, \ldots.$$

Now let u^* be an accumulation point of $(u^*_k)_{k=1}^\infty$ and v^* an accumulation point of $(v^*_k)_{k=1}^\infty$. As before $u^*(u) = 1$ and $v^*(v) = 1$. We also have, since

$$(u^*_k + v^*_k)(x_k/\|x_k\|) \le 2d(a,0)\|x_k\|^{-1},$$

that

$$(u^* + v^*)u \le 0.$$

This implies that $v^*(u) = -1$ and hence $v^*(v-u) = 2 \le \|v-u\|$. By strict convexity $v = -u$ which is impossible since $K \cap (-K) = \emptyset$. ∎

Now consider the cone $K \subset \ell_2^n$ of all $\xi = (\xi_1, \ldots, \xi_n)$ where $\xi_j \ge 0$ for $1 \le j \le n$. All subsets of K have the Lipschitz $\mathcal{C}$-UIEP by Theorem 6.7. But ℓ_2^n fails to have the collinearity property by Theorem 6.4 so that there is a sequence $(x_n)_{n=1}^\infty$ and $\epsilon > 0$ so that no three points are ϵ-collinear. Infinitely many of the $(x_n)_{n=1}^\infty$ must belong to one of the cones $K_{\theta_1,\ldots,\theta_n} = \{\xi : \theta_j\xi_j \ge 0,\ 1 \le j \le n\}$ for some choice of $\theta_j = \pm 1$ whence it follows that K also fails the collinearity property.

Theorem 6.8: *$M \oplus_1 M$ has the Lipschitz $\mathcal{C}$-IEP if and only if M has the collinearity property.*

Proof: If M has the collinearity property then $M \oplus_1 M$ also has the collinearity property (Proposition 6.2) and hence the Lipschitz $\mathcal{C}$-IEP (Theorem 6.5).

Now suppose $M \oplus_1 M$ has the Lipschitz $\mathcal{C}$-IEP. Suppose $\epsilon > 0$. Suppose $(x_n)_{n=1}^\infty$ is any sequence in M. We consider the sequences in $M \oplus_1 M$ defined by $u_n = (x_n, x_1)$ and $v_n = (x_1, x_n)$. Let $b = (x_1, x_1)$. By Theorem 5.2 (ii) we can find a fixed element $e = u_j$ or $e = v_j$ and an infinite subset $\mathbb{A}$ of $\mathbb{N}$ so that

$$d(u_n, e) + d(v_n, e) < d(u_n, b) + d(v_n, b) + \epsilon \quad n \in \mathbb{A}.$$

Hence

$$d(x_n, x_j) + d(x_n, x_1) + d(x_j, x_1) < 2d(x_n, x_1) + \epsilon \quad n \in \mathbb{N}.$$

This implies that

$$d(x_n, x_j) + d(x_j, x_1) < d(x_n, x_1) + \epsilon \quad n > j,\ n, j \in \mathbb{A}$$

so that M has the collinearity property. ∎

Example: It follows that if X is a finite-dimensional space which is not polyhedral then $X \oplus_1 X$ fails to have the Lipschitz $\mathcal{C}$-IEP. Thus the 4-dimensional space $\ell_2^2 \oplus_1 \ell_2^2$ fails to have the $\mathcal{C}$-IEP; this example was given by Lancien and Randrianantoanina.

It is however possible to give a 3-dimensional counter-example. Take a 2-dimensional space E which is not polyhedral, and use the Corollary to Theorem 7.7 of [11] to create a 3-dimensional space $X \supset E$ so that (X, E) fails the linear $\mathcal{C}$-IEP and hence also the Lipschitz $\mathcal{C}$-IEP by Lemma 2.2.

7. Finite-dimensional normed spaces with the Lipschitz $\mathcal{C}$-IEP

Although we have characterized finite-dimensional normed spaces with the Lipschitz $\mathcal{C}$-UIEP, the case of the Lipschitz $\mathcal{C}$-IEP is rather more subtle. We show here that in addition to polyhedral spaces (as proved by Lancien and Randrianantoanina [9]) this property is enjoyed by spaces with a Gateaux smooth norm and 2-dimensional spaces.

Suppose X is a finite-dimensional normed space, let D_X be the set of all $x \in \partial B_X$ so that the norm is Gateaux differentiable at x. The set D_X is dense in ∂B_X. Let D_X^* be the set of $x^* \in X^*$ with $\|x^*\| = 1$ and such that $x^*(x) = 1$ for some $x \in D_X$. Let $\partial_s B_{X^*}$ be the closure of D_X^*. Then $\partial_s B_{X^*}$ is the minimal closed boundary for X, i.e. the smallest closed subset G of B_{X^*} so that

$$\|x\| = \max\{x^*(x) : x \in G\}.$$

We shall say that X is **quasi-smooth** if

(QS1) If $0 \neq x \in X$ the set $\{x^* \in \partial_s B_{X^*} : x^*(x) = \|x\|\}$ is finite.

(QS2) For every $x_0^*, x_1^* \in \partial_s B_{X^*}$ with $x_1^* \neq x_0^*$ there is a neighborhood V of x_0^* so that if $x^* \in V$ and x is such that $x^*(x) = \|x\|$ then $x_0^*(x) \geq x_1^*(x)$.

Let us explain condition (QS2). For $x \in X$ define

$$p_x(y) = \max\{x^*(y) : x^*(x) = \|x\|, x^* \in \partial_s B_{X^*}\}.$$

LEMMA 7.1: *Let X be a finite-dimensional normed space with property (QS1). Then X satisfies (QS2) if and only if whenever $(u_n)_{n=1}^\infty$ is a sequence in X*

converging to some $u \neq 0$ *and* $(u_n^*)_{n=1}^\infty$ *is a sequence in* $\partial_s B_{X^*}$ *converging to some* u^* *and such that* $u_n^*(u_n) = \|u_n\|$ *for all* n*, then there exists* N *so that*

$$p_u(u_n) = u^*(u_n) \quad n \geq N.$$

Proof: First assume the condition of the lemma holds. If (QS2) fails for some pair (x_0^*, x_1^*) we can find a sequence (u_n^*) in $\partial_s B_{X^*}$ converging to x_0^* and a sequence (u_n) in ∂B_X such that $u_n^*(u_n) = 1$ and $x_0^*(u_n) < x_1^*(u_n)$. By passing to a subsequence we can suppose $(u_n)_{n=1}^\infty$ converges to some u and then $x_0^*(u) = 1 = x_1^*(u)$. Hence $p_u(u_n) = x_0^*(u_n)$ for n large enough, but $p_u(u_n) \geq x_1^*(u_n)$.

For the converse direction suppose instead that we have $(u_n)_{n=1}^\infty$ and $(u_n^*)_{n=1}^\infty$ as specified but $u^*(u_n) < p_u(u_n)$ for all n. Then by passing to a subsequence and using (QS1) we can assume there exists $v^* \neq u^*$ so that $v^* \in \partial_s B_{X^*}$, $v^*(u) = \|u\|$ and $p_u(u_n) = v^*(u_n)$ for all n. Then the pair (u^*, v^*) violates (QS2). ∎

PROPOSITION 7.2: *Suppose* X *is a finite-dimensional normed space. Then* X *is quasi-smooth if one of the following conditions holds:*

(i) X *is polyhedral;*
(ii) X *is Gateaux smooth;*
(iii) $\dim X = 2$.

Proof: If X is polyhedral then $\partial_s B_{X^*}$ is a finite set and the proof is trivial.

In case (ii), (QS1) is immediate. For (QS2) we argue that if $(u_n)_{n=1}^\infty$ is a convergent sequence in ∂B_X, with limit u, and $(u_n^*)_{n=1}^\infty$ is a convergent sequence in $\partial_s B_{X^*}$ with limit u^* and if $u_n^*(u_n) = 1$ then u^* norms u and $p_u(x) = u^*(x)$ so that Lemma 7.1 applies.

If $\dim X = 2$ we must first show that (QS1) holds. Suppose $u \in \partial B_X$ and $u^* \in \partial_s B_{X^*}$ norms u. Pick any v linearly independent of u. There is a sequence $u_n^* \in D_X^*$ and a sequence $u_n = s_n u + t_n v$ in D_X with $u_n^*(u_n) = 1$ and $\lim_{n\to\infty} u_n^* = u^*$. We can suppose u_n converges to some $w = su + tv$ where $s \geq 0$. The function $f(t) = \|u + tv\|$ is convex. If $t \neq 0$ then $u \neq w$ then $\|tw + (1-t)u\| = 1$ for $0 \leq t \leq 1$ and hence $u^*(v)$ is either the left or right-derivative of f at 0. If $t = 0$ then $s = 1$ and the properties of convex functions again show that $u^*(v)$ is either the left- or right- derivative of f at 0. This implies there are only two points in $\{u^* \in \partial_s B_{X^*} : u^*(u) = 1\}$.

We must also check (QS2); we again use Lemma 7.1. Again suppose (u_n) in ∂B_X and (u_n^*) in $\partial_s B_{X^*}$ satisfy $u_n^*(u_n) = 1$ and converge to u, u^* respectively. If

u admits only one norming functional in $\partial_s B_{X^*}$ then there is nothing to prove. Suppose it admits two, i.e. u^* and $w^* \neq u^*$. Defining v as before we see that

$$p_u(su + tv) = \max(s + tu^*(v), s + tw^*(v)).$$

For convenience we assume $u^*(v) > w^*(v)$ so that $u^*(v) = f'_+(0)$ and $w^*(v) = f'_-(0)$ where f'_+ and f'_- denote the right- and left-derivatives. If $u_n = s_n u + t_n v$ by the above reasoning $u_n^*(v) = s_n f'_\pm(t_n s_n^{-1})$ Since $s_n \to 1$ and $t_n \to 0$ from properties of convex functions we must have $t_n \geq 0$ eventually and then $p_u(u_n) = u^*(u_n)$. ∎

Theorem 7.3: *If X is quasi-smooth then X has the Lipschitz $\mathcal{C}$-IEP.*

Notice this gives a further proof of the Lancien-Randrianantoanina result when X is polyhedral but also establishes the result for the case when $\dim X = 2$ or X is Gateaux smooth. One can also easily visualize other examples.

Proof: We will verify Theorem 5.2; we note that we can assume $a = 0$ in (ii). Suppose for some $\epsilon > 0$ we can find two sequences $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty$ so that

$$\|x_n\| + \|y_n\| < \|x_n - x_k\| + \|y_n - x_k\| - \epsilon \quad 1 \leq k \leq n-1 \tag{7.1}$$

$$\|x_n\| + \|y_n\| < \|x_n - y_k\| + \|y_n - y_k\| - \epsilon \quad 1 \leq k \leq n-1. \tag{7.2}$$

Our proof will in fact only use (7.1). We must have

$$\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = \infty$$

(see the Remarks following Theorem 5.2).

By passing to subsequences we may suppose that $\|x_n\|, \|y_n\| > 0$ and

$$\lim_{n\to\infty} \frac{x_n}{\|x_n\|} = u, \quad \lim_{n\to\infty} \frac{y_n}{\|y_n\|} = v$$

where $\|u\| = \|v\| = 1$.

Let F be the (finite) set of $u^* \in \partial_s B_{X^*}$ so that $u^*(u) = 1$. Let F_0 be the subset of F of all u^* which are extreme in the convex hull of F. Define

$$p_u(x) = \max\{u^*(x) : u^* \in F\} = \max\{u^*(x) : u^* \in F_0\}.$$

We note that by passing to a further subsequence we can suppose that for some fixed $u_0^* \in F_0$ we have

$$u_0^*(x_k) = p_u(x_k) \quad k = 1, 2, \ldots.$$

Let $F_1 = F \setminus \{u_0^*\}$. Then it is clear that u_0^* is not in the convex hull of $F_1 \cup \{0\}$. Hence we can find a vector $z \in X$ so that

$$u_0^*(z) = 1 + \max\{u^*(z) : u^* \in F_1 \cup \{0\}\}.$$

Let $x_{n,k}^*$ be a norming functional in $\partial_s B_{X^*}$ for $x_n - x_k$, i.e. $\|x_{n,k}^*\| = 1$ and $x_{n,k}^*(x_n - x_k) = \|x_n - x_k\|$. Let $y_{n,k}^*$ be a norming functional in $\partial_s B_{X^*}$ for $y_n - x_k$.

By passing to an appropriate subsequence we can suppose that

$$\lim_{n\to\infty} x_{n,k}^* = x_k^*, \quad \lim_{n\to\infty} y_{n,k}^* = y_k^*,$$

It is clear that each x_k^* norms u and each y_k^* norms v and each belongs to $\partial_s B_{X^*}$.

Now since X is quasi-smooth the sets

$$\{x_k^* : k = 1, 2, \ldots\} \quad \text{and} \quad \{y_k^* : k = 1, 2, \ldots\}$$

are finite and by passing to a further subsequence we can assume that they are constant. Let us put $x_k^* = u_1^*$ for all k and $y_k^* = v^*$ for all k.

We further have (from Lemma 7.1) that for each k there exists $N_0(k)$ so that

$$u_1^*(x_n - x_k) = p_u(x_n - x_k) \quad n > N_0(k).$$

We now may pass to a subsequence and further assume that

$$u_1^*(x_n - x_k) = p_u(x_n - x_k) \quad n > k.$$

We also have

$$x_{n,k}^*(x_n) + y_{n,k}^*(y_n) < x_{n,k}^*(x_n - x_k) + y_{n,k}^*(y_n - x_k) - \epsilon$$

which simplifies to

$$(x_{n,k}^* + y_{n,k}^*)(x_k) < -\epsilon.$$

Letting $n \to \infty$ we obtain

$$(u_1^* + v^*)(x_k) \leq -\epsilon \quad k = 1, 2, \ldots.$$

Thus

$$(u_1^* + v^*)\Big(\frac{x_k}{\|x_k\|}\Big) \leq 0$$

so that

$$(u_1^* + v^*)(u) \leq 0.$$

Hence $v^*(u) = -1$ and so $-v^*$ also norms u. It is clear that $u_1^* + v^* \neq 0$ (so that if X is Gateaux smooth we have a contradiction and the proof is complete).

In the general case we also have $-v^*(x_k) \leq u_0^*(x_k)$ so that

$$(u_1^* - u_0^*)(x_k) \leq -\epsilon, \quad k = 1, 2, \ldots.$$

We also have

$$u_1^*(x_n - x_k) = p_u(x_n - x_k) \geq u_0^*(x_n - x_k) \quad k < n.$$

Hence

$$(u_1^* - u_0^*)(x_n - x_{n-1}) \geq 0 \quad n = 2, 3 \ldots.$$

Thus the sequence $(u_1^* - u_0^*)(x_k)$ is monotone increasing and bounded above. We conclude that $\lim_{k\to\infty}(u_1^* - u_0^*)(x_k)$ exists. Let

$$\delta_k = \sup_{n>k}(u_1^* - u_0^*)(x_n - x_k) \quad k = 1, 2, \ldots$$

so that $\lim_{k\to\infty} \delta_k = 0$.

For any $n > k$ let us consider

$$w_{n,k} = x_n - x_k + ((u_1^* - u_0^*)(x_n - x_k))z + 2^{-n}z.$$

Then

$$\lim_{n\to\infty} \frac{w_{n,k}}{\|w_{n,k}\|} = u.$$

Recall that if $u^* \neq u_0^*$ and $u^* \in F$ then

$$u^*(z) \leq u_0^*(z) - 1.$$

Hence

$$\begin{aligned} u^*(w_{n,k}) &< \langle x_n - x_k, u^* - u_1^* + u_0^* \rangle + \langle x_n - x_k, u_1^* - u_0^* \rangle u_0^*(z) + 2^{-n} u_0^*(z) \\ &\leq u_0^*(w_{n,k}). \end{aligned}$$

Thus if $w_{n,k}^* \in \partial_s B_{X^*}$ norms $w_{n,k}$ we have (using quasi-smoothness, in particular Lemma 7.1)

$$\lim_{n\to\infty} w_{n,k}^* = u_0^*.$$

Now appealing to (7.1)

$$\|x_n\| + \|y_n\| \leq \|w_{n,k}\| + \|w_{n,k} - (x_n - x_k)\| + \|y_n - x_k\| - \epsilon$$

and hence

$$w^*_{n,k}(x_n - w_{n,k}) + y^*_{n,k}(x_k) \le \|w_{n,k} - (x_n - x_k)\| - \epsilon$$

which reduces to

$$w^*_{n,k}(x_k) + y^*_{n,k}(x_k) \le 2(2^{-n} + \delta_k)\|z\| - \epsilon.$$

Taking limits as $n \to \infty$ we have

$$(u_0^* + v^*)(x_k) \le 2\delta_k\|z\| - \epsilon.$$

However $u_0^*(x_k) \ge -v^*(x_k)$ for all k so that

$$0 < \epsilon \le 2\delta_k\|z\| \quad k = 1, 2 \ldots .$$

This contradiction proves the Theorem. ∎

8. Hölder extensions

We will now consider a problem raised by Naor [14]. Suppose for a given Banach space Y, M is a metric space with the Y-IEP. Naor asks if for $0 < \alpha < 1$ it is always true that for any subset E of M and any α-Hölder continuous map $F_0\colon E \to Y$ satisfying

$$\|F_0(x) - F_0(y)\| \le d(x,y)^\alpha \quad x, y \in E$$

one has an extension $F\colon M \to Y$ with

$$\|F(x) - F(y)\| \le d(x,y)^\alpha \quad x, y \in M.$$

Naor conjectures that is false for arbitrary Y but true for Y a Hilbert space. We shall resolve this question for c_0 and $\mathcal{C}(K)$-spaces where K is compact metric and, in particular, show that it is false in the latter case.

If M is a metric space and $0 < \alpha < 1$ we denote by M^α the metric space (M, d^α). Naor's question is then whether M^α has Lipschitz Y-IEP whenever M has the Lipschitz Y-IEP and $0 < \alpha < 1$.

Proposition 8.1: *Suppose M has the Lipschitz c_0-AIEP. Then for every $0 < \alpha < 1$, M^α has the Lipschitz c_0-IEP.*

Proof: If not there exists $0 < \alpha < 1$, $\epsilon > 0$, $a \in M$ and a sequence $(x_n)_{n=1}^\infty$ such that

$$d(x_n, a)^\alpha + \epsilon < d(x_n, x_j)^\alpha \quad j < n.$$

Note this implies that $(x_n)_{n=1}^\infty$ is metrically bounded. Indeed, if not, by passing to a subsequence we can assume that $d(x_n, a) \to \infty$. But then for every j we have

$$\lim_{n\to\infty} (d(x_n, x_j)^\alpha - d(x_n, a)^\alpha) = 0$$

which is impossible. However if $(x_n)_{n=1}^\infty$ is metrically bounded then we deduce the existence of $\eta > 0$ so that

$$d(x_n, a) + \eta < d(x_n, x_j) \quad j < n$$

which contradicts the fact that M has Lipschitz c_0-AIEP (Theorem 5.1). ∎

Proposition 8.2: *Suppose M is a metric space and $0 < \alpha < 1$. Then M^α has the Lipschitz $\mathcal{C}$-IEP (respectively, c_0-IEP) if and only if M^α has the Lipschitz $\mathcal{C}$-AIEP (respectively, c_0-AIEP).*

Proof: Suppose M^α has Lipschitz $\mathcal{C}$-AIEP but not Lipschitz $\mathcal{C}$-IEP. Then, by Theorem 5.2 we can find $a \in M$, $\epsilon > 0$ and two sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ such that

$$d(x_n, a)^\alpha + d(y_n, a)^\alpha + \epsilon < d(x_n, x_j)^\alpha + d(y_n, x_j)^\alpha \quad j < n \tag{8.1}$$

and

$$d(x_n, a)^\alpha + d(y_n, a)^\alpha + \epsilon < d(x_n, y_j)^\alpha + d(y_n, y_j)^\alpha \quad j < n. \tag{8.2}$$

Let us suppose one of $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty$, say $(x_n)_{n=1}^\infty$, is not metrically bounded; by passing to a subsequence we can suppose $d(x_n, a) \to \infty$ and, as in the preceding Proposition,

$$\lim_{n\to\infty} (d(x_n, y_j)^\alpha - d(x_n, a)^\alpha) = 0 \quad j < n.$$

It is thus possible to pass to a further subsequence such that

$$d(y_n, a)^\alpha + \frac{1}{2}\epsilon < d(y_n, y_j)^\alpha \quad j < n.$$

If $(y_n)_{n=1}^\infty$ is metrically bounded this contradicts the fact that M^α has the $\mathcal{C}$-AIEP (and hence the c_0-AIEP). If it is not metrically bounded then we can apply a similar argument to the above to deduce that

$$\liminf_{n\to\infty} (d(y_n, y_j)^\alpha - d(y_n, a)^\alpha) = 0$$

and hence get a contradiction.

If both $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are metrically bounded then we have a contradiction to the fact that M^α has Lipschitz $\mathcal{C}$-AIEP. ∎

PROPOSITION 8.3: *Suppose M has the Heine-Borel property. Then for every $0 < \alpha < 1$, M^α has Lipschitz $\mathcal{C}$-IEP.*

Proof: This follows from Propositions 8.2 and Proposition 4.5. ∎

Note that this Proposition implies that for any finite-dimensional normed space X we have isometric Hölder continuous extensions into $\mathcal{C}(K)$-spaces; this result is due to Lancien and Randrianantoanina [9].

We recall (see [1]) that a metric space M is **stable** if for any pair of metrically bounded sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ we have

$$\lim_{m\to\infty}\lim_{n\to\infty} d(x_m,y_n) = \lim_{n\to\infty}\lim_{m\to\infty} d(x_m,y_n)$$

whenever all the limits exist. Any subset of an L_p-space when $1 \le p < \infty$ is stable.

THEOREM 8.4: *Let M be a stable metric space with the Lipschitz $\mathcal{C}$-AIEP; then for every $0 < \alpha < 1$, M^α has the Lipschitz $\mathcal{C}$-IEP.*

Proof: Suppose M^α fails the Lipschitz $\mathcal{C}$-IEP for some $0 < \alpha < 1$. Then it also fails Lipschitz $\mathcal{C}$-AIEP by Proposition 8.2. Hence by Theorem 5.2 we can find $a \in M$, $\epsilon > 0$ and metrically bounded sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ such that (8.1) and (8.2) hold.

We may suppose by passing to further subsequences that the following limits exist:

$$\begin{aligned}
\lim_{n\to\infty} d(x_n,a) &= \delta_{xa},\\
\lim_{n\to\infty} d(y_n,a) &= \delta_{ya},\\
\lim_{n\to\infty}\lim_{m\to\infty} d(x_m,x_n) &= \delta_{xx},\\
\lim_{n\to\infty}\lim_{m\to\infty} d(y_m,y_n) &= \delta_{yy},\\
\lim_{n\to\infty}\lim_{m\to\infty} d(x_m,y_n) &= \delta_{xy},\\
\lim_{n\to\infty}\lim_{m\to\infty} d(y_m,x_n) &= \delta_{yx}.
\end{aligned}$$

Stability of M implies that $\delta_{xy} = \delta_{yx}$.

Since M has the Lipschitz $\mathcal{C}$-AIEP it follows that either

$$\delta_{xx} + \delta_{yx} \le \delta_{xa} + \delta_{ya} \tag{8.3}$$

or

$$\delta_{yy} + \delta_{xy} \le \delta_{xa} + \delta_{ya}. \tag{8.4}$$

Let us assume the former.

By (8.1) we have

$$\delta_{xa}^{\alpha} + \delta_{ya}^{\alpha} < \delta_{xx}^{\alpha} + \delta_{yx}^{\alpha}. \tag{8.5}$$

We also note that since M also has Lipschitz c_0-AIEP we also have

$$\delta_{xx} \leq \delta_{xa},\ \delta_{yy} \leq \delta_{ya}. \tag{8.6}$$

Combining (8.5) and (8.3) and using the fact that $0 < \alpha < 1$ it is clear that one of $\{\delta_{xa}, \delta_{ya}\}$ is greater than or equal to both δ_{xx} and δ_{yx} and one is less than or equal to both. From (8.6) it follows that we cannot have $\delta_{xa} \leq \min(\delta_{xx}, \delta_{yx})$ as this would imply that $\delta_{xx} = \delta_{xa}$ and render (8.5) and (8.3) impossible simultaneously. Hence

$$\delta_{ya} \leq \min(\delta_{xx}, \delta_{yx}) \leq \max(\delta_{xx}, \delta_{yx}) \leq \delta_{xa}.$$

But now by (8.6) we have

$$\delta_{yy} \leq \delta_{ya} \leq \delta_{xx}.$$

By stability this means that (8.4) also holds, which implies by similar reasoning that we also have:

$$\delta_{xa} \leq \min(\delta_{yy}, \delta_{xy}) \leq \max(\delta_{yy}, \delta_{xy}) \leq \delta_{ya}.$$

Thus $\delta_{xa} = \delta_{xx} = \delta_{yx} = \delta_{ya}$ which is absurd. ∎

It is now possible to give an example to show that Naor's problem has a negative solution.

Example: There is a metric space M with the property that M has Lipschitz $\mathcal{C}$-IEP but M^{α} fails Lipschitz $\mathcal{C}$-AIEP for every $0 < \alpha < 1$.

We defined M to be subset of $\mathbb{Z}^2$ consisting of all pairs (m, n) with $m \in \mathbb{N}$ and $n = 0, 1$ together with the origin $(0, 0)$. Let us define a metric on M by

$$\begin{aligned} d((m_1,0),(m_2,0)) &= 3 \quad m_1 \neq m_2, \\ d((m_1,1),(m_2,1)) &= 2 \quad m_1 \neq m_2, \\ d((m_1,0),(m_2,1)) &= 3 \quad m_1 \leq m_2, \\ d((m_1,0),(m_2,1)) &= 5 \quad m_1 > m_2, \\ d((0,0),(m,0)) &= 4 \quad m \in \mathbb{N} \\ d((0,0),(m,1)) &= 2 \quad m \in \mathbb{N}. \end{aligned}$$

We first need to verify that M has the Lipschitz $\mathcal{C}$-IEP. Indeed if not we can suppose the existence of $a \in M$, $\epsilon > 0$ and sequences $(x_n)_{n=1}^\infty, (y_n)_{n=1}^\infty$ so that (5.2) and (5.3 hold for $\lambda = 1$). By passing to a subsequence we can suppose that $(0,0)$ is not in either sequence and the second co-ordinate of both sequences is constant. We then have (taking into account the fact that all distances are integers),

$$d(x_n, a) + d(y_n, a) + 1 \leq d(x_n, x_j) + d(y_n, x_j) \quad j < n$$

and

$$d(x_n, a) + d(y_n, a) + 1 \leq d(x_n, y_j) + d(y_n, y_j) \quad j < n.$$

It is then simple but tedious to check cases. If the second co-ordinate for both sequences is 0 the right-hand sides tend to 6 as $n \to \infty$ but the left-hand side tends to a limit no less than 7. If the second co-ordinate is 1 for both sequences the right-hand sides tend to 4 but the left-hand side tends to a limit no less than 5. If say the $(x_n)_{n=1}^\infty$ all have second co-ordinate 0 and the $(y_n)_{n=1}^\infty$ all have second co-ordinate 1 the right-hand limits are 6 and 7. However the left-hand limits are both at least 7. Thus M has the Lipschitz $\mathcal{C}$-IEP.

Let us check however that for every $0 < \alpha < 1$, M^α fails Lipschitz $\mathcal{C}$-IEP. Indeed let $a = (0,0)$, $x_n = (n,0)$ and $y_n = (n,1)$. Then

$$d(x_n, a)^\alpha + d(y_n, a)^\alpha = 2^\alpha + 4^\alpha \quad n \in \mathbb{N}$$
$$d(x_n, x_j)^\alpha + d(y_n, x_j)^\alpha = 2.3^\alpha > 2^\alpha + 4^\alpha \quad j < n$$

and

$$d(x_n, y_j)^\alpha + d(y_n, y_j)^\alpha = 2^\alpha + 5^\alpha > 2^\alpha + 4^\alpha \quad j < n.$$

References

[1] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis. Vol. 1*, American Mathematical Society Colloquium Publications, vol. 48, American Mathematical Society, Providence, RI, 2000.

[2] K. Borsuk, *Über Isomorphic der Funktionalraüme*, Bulletin of the Institute Ac. Pol. Sci. **A113** (1933), 1–10.

[3] J. A. C. Burlak, R. A. Rankin and A. P. Robertson, *The packing of spheres in the space l_p*, Proceedings of the Glasgow Mathematical Association **4** (1958), 22–25.

[4] J. Elton and E. Odell, *The unit ball of every infinite-dimensional normed linear space contains a* $(1+\varepsilon)$*-separated sequence*, Colloquium Mathematicum **44** (1981), 105–109.

[5] R. Espínola and G. López, *Extension of compact mappings and* $\aleph_0$*-hyperconvexity,* Nonlinear Analysis **49** (2002), 1127–1135.

[6] V. Klee, *Some characterizations of convex polyhedra,* Acta Mathematica **102** (1959), 79–107.

[7] C. A. Kottman, *Packing and reflexivity in Banach spaces,* Transactions of the American Mathematical Society **150** (1970), 565–576.

[8] A. Kryczka and S. Prus, *Separated sequences in nonreflexive Banach spaces,* Proceedings of the American Mathematical Society **129** (2001), 155–163.

[9] G. Lancien and B. Randrianantoanina, *On the extension of Hölder maps with values in spaces of continuous functions,* Israel Journal of Mathematics **147** (2005), 75–92.

[10] J. Lindenstrauss, *On nonlinear projections in Banach spaces,* Michigan Mathematical Journal **11** (1964), 263–287.

[11] J. Lindenstrauss, *Extension of compact operators,* Memoirs of the American Mathematical Society **48** (1948), 112.

[12] J. Melleray, *On the geometry of Urysohn's universal metric space,* Topology and its Applications **154** (2007), 384–403.

[13] A. A. Miljutin, *Isomorphisms of the spaces of continuous functions over compact sets of the cardinality of the continuum,* Teoriya Funkciĭ Funkcional. Anal. i Priložen. Vyp. **2** (1966), 150–156 (1 foldout) (Russian).

[14] A. Naor, *A phase transition phenomenon between the isometric and isomorphic extension problems for Hölder functions between* L_p *spaces,* Mathematika **48** (2001), 253–271 (2003).